Mathematics
Class XII
Sample Paper – 9 Solution

SECTION – A

1. a_{42}, means element at 3rd row and 2nd column

So,

$a_{32} = 10$

2. Differentiating w.r.t. x, we get,

$$\frac{d}{dx}(\sin(\cos x))$$
$$= \cos(\cos x)\frac{d}{dx}(\cos x)$$
$$= -\sin x \cos(\cos x)$$

3. DE:

$$x + \left(\frac{dy}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

squaring

$$x^2 + 2x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$
$$x^2 + 2x\left(\frac{dy}{dx}\right) = 1$$

It is linear, since x is independent variable.

4. Let θ be the angles between, the given two lines

So, the angle between them given their direction cosines is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

substituting we get

$$\theta = \cos^{1}\left(\frac{8}{5\sqrt{3}}\right)$$

OR

Let θ be the angles between, the given two lines

So, the angle between them given their direction cosines is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

substituting we get

$$\theta = \cos^{1}\left(\frac{10}{9\sqrt{22}}\right)$$

SECTION – B

5. f: $R_+ \to [4, \infty)$ defined by $f(x) = x^2 + 4$

f is 1 – 1

Let $x_1, x_2, \in R_+$ such that $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 + 4 = x_2^2 + 4$$
$$\Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1 = x_2 \qquad \because x_1, x_2 \in R_+$$

∴ f is 1 – 1

f is onto: Let $y \in [4, \infty)$

$f(x) = y \Rightarrow x^2 + 4 = y$

$$\Rightarrow x = \sqrt{y-4}$$

Since $y \in [4, \infty) \Rightarrow x \in R_+$

For $y \in [4, \infty)$ there is a $x \in R_+$ such that $f(x) = y$.

So f is onto.

So, f is bijective function and hence f is invertible.

The inverse of f is defined by

f: $[4, \infty) \to R_+$

$f^{-1}(y) = \sqrt{y-4}$

6.

We have,

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix} \text{ and, } B=\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$A+B=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -1\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}=\text{diag}(3\ 2\ 1)$$

and,

$$3A+4B=\begin{bmatrix}3 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 6\end{bmatrix}+\begin{bmatrix}8 & 0 & 0\\ 0 & 12 & 0\\ 0 & 0 & -4\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 2\end{bmatrix}=\text{diag}(11\ 9\ 2)$$

7. $I=\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$

Let $e^x=t \quad e^x dx=dt$

Now integral I becomes,

$$I=\int \frac{dt}{\sqrt{5-4t-t^2}}$$

$$\Rightarrow I=\int \frac{dt}{\sqrt{5+4-4-4t-t^2}}$$

$$\Rightarrow I=\int \frac{dt}{\sqrt{9-\left(4+4t+t^2\right)}}$$

$$\Rightarrow I=\int \frac{dt}{\sqrt{9-(t+2)^2}}$$

$$\Rightarrow I=\int \frac{dt}{\sqrt{3^2-(t+2)^2}}$$

$$\Rightarrow I=\sin^{-1}\frac{(t+2)}{3}+C$$

$$\Rightarrow I=\sin^{-1}\frac{(e^x+2)}{3}+C$$

8. $I=\int \frac{(x-4)e^x}{(x-2)^3}.dx$

$$I=\int e^x\left(\frac{x-2}{(x-2)^3}-\frac{2}{(x-2)^3}\right).dx$$

$$I=\int e^x\left(\frac{1}{(x-2)^2}-\frac{2}{(x-2)^3}\right).dx$$

Thus the given integral is of the form,

$$I=\int e^x\left|f(x)+f'(x)\right|dx \text{ where, } f(x)=\frac{1}{(x-2)^2};\ f'(x)=\frac{-2}{(x-2)^3}$$

$$I=\int \frac{e^x}{(x-2)^2}dx-\int \frac{2e^x}{(x-2)^3}dx$$

$$=\int e^x\left[\frac{1}{(x-2)^2}+\frac{d}{dx}\left(\frac{1}{(x-2)^2}\right)\right]dx$$

$$\text{So}, I=\frac{e^x}{(x-2)^2}+C$$

OR

$$\int \frac{x^2}{1+x^3}dx$$

Let $1 + x^3 = t$

$\Rightarrow 0 + 3x^2dx = dt$

$$\Rightarrow x^2dx=\frac{dt}{3}$$

$$\therefore \int\left(\frac{x^2}{1+x^3}\right)dx=\int \frac{\frac{dt}{3}}{t}$$

$$=\frac{1}{3}\int \frac{dt}{t}$$

$$=\frac{1}{3}\log|t|+c$$

$$=\frac{1}{3}\log\left|1+x^3\right|+c$$

9. We have to differentiate it w.r.t. x two times

Differentiating

$$2y\frac{dy}{dx}=m(-2x)$$

$$y\frac{dy}{dx}=-mx...............(1)$$

differentiating again

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=-m$$

from (1)

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=\frac{y}{x}\frac{dy}{dx}$$

which is the required differential equation

10.

$$\vec{r}.\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$$

$$\Rightarrow 3x+4y-12z+13=0$$

Distance of point (1, 1, p) from the plane , is given by

$$\frac{|3\times1+4\times1-12\times p+13|}{\sqrt{3^2+4^2+(-12)^2}}=\frac{|20-12p|}{\sqrt{3^2+4^2+(-12)^2}}$$

Distance of point (-3, 0, 1) from the plane , is given by

$$\frac{|3\times(-3)+4\times0-12\times1+13|}{\sqrt{3^2+4^2+(-12)^2}}=\frac{|-8|}{\sqrt{3^2+4^2+(-12)^2}}$$

The two distances are equal

$$\Rightarrow\frac{|20-12p|}{\sqrt{3^2+4^2+(-12)^2}}=\frac{|-8|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$\Rightarrow|20-12p|=|-8|$$

$$\Rightarrow 20-12p=\pm8$$

$$\Rightarrow p=1,\frac{7}{3}$$

OR

$$\text{L.H.S.} = [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$$
$$= \left(\vec{a}+\vec{b}\right).\left[\left(\vec{b}+\vec{c}\right)\times\left(\vec{c}+\vec{a}\right)\right]$$
$$= \left(\vec{a}+\vec{b}\right).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$
$$= \left(\vec{a}+\vec{b}\right).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}]$$
$$= \vec{a}.(\vec{b}\times\vec{c})+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{c}\times\vec{a}\right)+\vec{b}.(\vec{b}\times\vec{c})+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$= \vec{a}.(\vec{b}\times\vec{c})+0+0+0+0+\vec{b}.\left(\vec{c}\times\vec{a}\right)$$
$$= 2\vec{a}.(\vec{b}\times\vec{c})$$
$$= 2[\vec{a}\ \vec{b}\ \vec{c}] = \text{R.H.S.}$$

11. Let p be the probability that a student entering a university graduates,

So p = 0.4.

Since p + q = 1, thus, q = 1 - 0.4 = 0.6.

Let X denote the random variable representing the number of students who graduate out of the 3. Probability that r students graduate out of n entering the university is given by

$$P\left(X=r\right) = {}^{n}C_{r}p^{r}q^{n-r}$$
$$= {}^{3}C_{r}\left(0.4\right)^{r}\left(0.6\right)^{n-r} \quad \text{....(1)}$$

(i)

Probability that none will graduate

$$= P\left(X = 0\right)$$
$$= {}^{3}C_{0}\left(0.4\right)^{0}\left(0.6\right)^{3-0}$$
$$= \left(0.6\right)^{3}$$
$$= 0.216$$

$\therefore$ Probability that none will graduate = 0.216

(ii)

$$\begin{aligned}
&\text{Probability that one will graduate}\\
&= P(X=1)\\
&= {}^3C_1(0.4)^1(0.6)^{3-1}\\
&= 3\times(0.4)\times(0.36)\\
&= 0.432\\
&\therefore \text{Probability that only one will graduate} = 0.432
\end{aligned}$$

12. Let E_1, E_2, and A be the events.

E_1 = white ball is transferred from 1st bag to 2nd bag.

E_2 = black ball is transferred from 1st bag to 2nd bag.

A = a black ball is drawn.

So, $P(E_1) = \frac{5}{8}$; $P(E_2) = \frac{3}{8}$;

Also, $P(A/E_1)$ = Probability of taking out black ball from bag 2 when white ball is already transferred from 1st bag to 2nd bag.

$\Rightarrow P(A/E_1) = \frac{4}{8}$

$P(A/E_2)$ = Probability of taking out black ball from bag 2 when black ball is transferred from 1st bag to 2nd bag.

$\Rightarrow P(A/E_2) = \frac{5}{8}$

By law of total probability

$$P(E_2 | A) = \frac{P(E_2).P(A|E_2)}{P(E_1).P(A|E_1) + P(E_2).P(A|E_2)}$$

$$= \frac{\frac{3}{8}.\frac{5}{8}}{\frac{5}{8}.\frac{4}{8}+\frac{3}{8}.\frac{5}{8}} = \frac{15}{35} = \frac{3}{7}$$

OR

The events A, E_1, E_2, E_3, and E_4 are given by

A = event when doctor visits patients late

E_1 = doctor comes by train

E_2 = doctor comes by bus

E_3 = doctor comes by scooter

E_4 = doctor comes by other means of transport

So, $P(E_1) = \frac{3}{10}$, $P(E_2) = \frac{1}{5}$, $P(E_3) = \frac{1}{10}$, $P(E_4) = \frac{2}{5}$

$P(A/E_1)$ = Probability that the doctor arrives late, given that he is comes by train = $\frac{1}{4}$

Similarly $P(A/E_2) = \frac{1}{3}$, $P(A/E_3) = \frac{1}{12}$, $P(A/E_4) = 0$

Required probability of the doctor arriving late by train by using Baye's theorem,

$$P(E_1/A) = \frac{P(E_1)P\left(\frac{A}{E_1}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+P(E_3)P\left(\frac{A}{E_3}\right)+P(E_4)P\left(\frac{A}{E_4}\right)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0}$$

$$= \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence the required probability is $\frac{1}{2}$.

SECTION – C

13. (i) $f: N \to Z$

$f(x) = x$

and f: $Z \to Z$ s.t.

$g(x) = |x|$

Clearly, g the absolute value function is not one–one.

gof: $N \to Z$

gof $(x) = |x| \ \forall \ x \in N$

$g(f(x)) = g(x) = |x|$

Let gof (x_1) =g of $((x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$|x_1| = |x_2|$

$\Rightarrow x_1 = x_2$

since $x_1, x_2 \in N$

So gof one-one.

(ii) $f: N \to N$ such that

$f(x) = x + 1$ and

g: $N \to N$ such that $g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$

Clearly f is not an onto function since set of natural numbers is infinite

gof: $N \to N$ s.t

gof $(x) = g(f(x)) = g(x + 1) = x$ since x is a natural number so $x + 1 > 1$

Now consider any natural number $x \in N$, the codomain set of gof,

Every natural number has a predecessor i.e every x in N can be mapped to $x - 1$ in N. So gof is onto.

OR

For one-one function

Let $x_1 = x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\frac{2x_1 - 1}{3} = \frac{2x_2 - 1}{3}$$
$$\Rightarrow 2x_1 - 1 = 2x_2 - 1$$
$$\Rightarrow x_1 = x_2$$

So, f is a 1–1 function

For onto function

Let $y \in R$ such that $f(x) = y$

$$\frac{2x - 1}{3} = y \Rightarrow 2x - 1 = 3y$$
$$\Rightarrow x = \frac{3y + 1}{2}$$
$$f\left(\frac{3y+1}{2}\right) = \frac{2\left(\frac{3y+1}{2}\right) - 1}{3}$$
$$= \frac{3y + 1 - 1}{3} = y$$

Therefore, the function f(x) is onto.

The function is bijective , therefore invertible.

For inverse function

Since, f(x) is one-one and onto, therefore

$f^{-1}(x)$ exists.

Let $y = f(x)$

$$y = \frac{2x - 1}{3}$$
$$\Rightarrow 3y = 2x - 1$$
$$\Rightarrow x = \frac{3y + 1}{2}$$
$$\therefore f^{-1}(y) = \frac{3y + 1}{2}$$

Hence, $f^{-1}(x) = \frac{3x + 1}{2}$

14. Let $\sin^{-1}x = \theta$ or $\sin\theta = x$

So given equation becomes

$$\sin^{-1}(1-x) - 2\theta = \frac{\pi}{2}$$

$$\sin^{-1}(1-x) = \frac{\pi}{2} + 2\theta$$

$$1 - x = \sin\left(\frac{\pi}{2} + 2\theta\right)$$

$$1 - x = \cos 2\theta$$

$$1 - x = 1 - 2\sin^2\theta$$

$$\Rightarrow 1 - x = 1 - 2x^2$$

$$\Rightarrow 2x^2 - x = 0$$

$$\Rightarrow x(2x - 1) = 0$$

$$\Rightarrow x = 0, \qquad x = \frac{1}{2}$$

If $x = 0$

L.H.S of given equation $= \sin^{-1}1 - 2\sin^{-1}0 = \frac{\pi}{2} =$ R.H.S.

If $x = \frac{1}{2}$

L.H.S. of given equation $= \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6} \neq$ R.H.S.

$\therefore$ solution is $x = 0$

15.

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1; R_3 \to R_3 - R_1$

$$\Rightarrow (1+xyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\Rightarrow (1+xyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x+y \\ 0 & 0 & z-y \end{vmatrix} = 0$$

$$\Rightarrow (1+xyz)(x-y)(y-z)(z-x) = 0$$

Given, x , y , z are different $\Rightarrow 1+xyz=0 \Rightarrow xyz = -1$

16.

$y = x^{x^x}$

then,

$y = e^{\log x^{x^x}}$

$y = e^{x^x \log x}$

differentiating w.r.t. x

$$\frac{dy}{dx} = e^{x^x \log x} \frac{d}{dx}\left(x^x \log x\right)$$

$$\frac{dy}{dx} = y\left(x^x \frac{d}{dx}\log x + \log x \frac{d}{dx} x^x\right)$$

$$\frac{dy}{dx} = y\left(x^x \frac{d}{dx}\log x + \log x \frac{d}{dx} e^{\log x^x}\right)$$

$$\frac{dy}{dx} = y\left(\frac{x^x}{x} + \log x \frac{d}{dx} e^{x\log x}\right)$$

$$\frac{dy}{dx} = y\left(\frac{x^x}{x} + \log x \frac{d}{dx} e^{x\log x}\right)$$

$$\frac{dy}{dx} = y\left(\frac{x^x}{x} + \log x\left(e^{x\log x} \frac{dy}{dx} x \log x\right)\right)$$

$$\frac{dy}{dx} = y\left(\frac{x^x}{x} + \log x\left(e^{x\log x}\left(x \times \frac{1}{x} + \log x\right)\right)\right)$$

$$\frac{dy}{dx} = y\left(\frac{x^x}{x} + \log x\left(x^x\left(1 + \log x\right)\right)\right)$$

$$\frac{dy}{dx} = x^{x^x} x^x\left(\frac{1}{x} + \log x\left(1 + \log x\right)\right)$$

OR

we have,

$$y=(\sin x)^{\tan x}+(\cos x)^{\sec x}$$

$$y=e^{\tan x\,.\log\sin x}+e^{\sec x\,.\log\cos x}$$

differentiating

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{\tan x\,.\log\sin x}\right)+\frac{d}{dx}\left(e^{\sec x\,.\log\cos x}\right)$$

$$\frac{dy}{dx}=e^{\tan x\,.\log\sin x}\frac{d}{dx}(\tan x\,.\log\sin x)+e^{\sec x\,.\log\cos x}\frac{d}{dx}(\sec x\,.\log\cos x)$$

$$\frac{dy}{dx}=(\sin x)^{\tan x}\left(\tan x\frac{d}{dx}\log\sin x+\log\sin x\frac{d}{dx}\tan x\right)+$$

$$(\cos x)^{\sec x}\frac{d}{dx}\left(\sec x\frac{d}{dx}\log\cos x+\log\cos x\frac{d}{dx}\sec x\right)$$

$$\frac{dy}{dx}=(\sin x)^{\tan x}\left(\sec^2 x.\log\sin x+1\right)+(\cos x)^{\sec x}\left(\sec x.\tan x.\log\cos x-\sec x.\tan x\right)$$

17. Let

$$y=x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$$

$$y=e^{\cot x.\log x}+\frac{2x^2-3}{x^2+x+2}$$

differentiate

$$\frac{dy}{dx}=e^{\cot x.\log x}\times\frac{d}{dx}(\cot x.\log x)+\frac{d}{dx}\left(\frac{2x^2-3}{x^2+x+2}\right)$$

$$\frac{dy}{dx}=x^{\cot x}\times\left(\cot x\times\frac{1}{x}+\log x\times-\operatorname{cosec}^2 x\right)+\frac{(x^2+x+2)4x-(2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

$$\frac{dy}{dx}=x^{\cot x}\times\left(\cot x\times\frac{1}{x}+\log x\times-\operatorname{cosec}^2 x\right)+\frac{2x^2+14x+3}{(x^2+x+2)^2}$$

18.

$$y=\frac{4\sin\theta}{2+\cos\theta}-\theta, 0\le\theta\le\pi,$$

$$\frac{dy}{dx}=\frac{(2+\cos\theta)(4\cos\theta)+4\sin\theta(\sin\theta)}{(2+\cos\theta)^2}-1$$

$$=\frac{8\cos\theta+4(\cos^2\theta+\sin^2\theta)-4-\cos^2\theta-4\cos\theta}{(2+\cos\theta)^2}$$

$$=\frac{4\cos\theta-\cos^2\theta}{(2+\cos\theta)^2}$$

$$=\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Since, $(2+\cos\theta)^2>0$ for all θ

$4-\cos\theta>0$ for all θ as $|\cos\theta|\le 1$

$\therefore \frac{dy}{d\theta}>0$ if $\cos\theta>0$

$\therefore$ If $\theta\in\left[0,\frac{\pi}{2}\right]$ $\frac{dy}{d\theta}>0$

$\therefore$ y is increasing in $\left[0,\frac{\pi}{2}\right]$

19.

$$\int \frac{x^2}{x^4+x^2-2}dx$$

$$=\int \frac{x^2}{x^2-1\ \ x^2+2}dx$$

$$=\int \frac{x^2}{x-1\ \ x+1\ \ x^2+2}dx$$

Using partial fraction,

$$\frac{x^2}{(x-1)(x+1)(x^2+2)}=\frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{Cx+D}{(x^2+2)}$$

$$\frac{x^2}{(x-1)(x+1)(x^2+2)}=\frac{A(x+1)(x^2+2)+B(x-1)(x^2+2)+\ Cx+D\ (x-1)(x+1)}{(x-1)(x+1)(x^2+2)}$$

Equating the coefficients from both the numerators we get,

$A+B+C=0........(1)$

$A-B+D=1........(2)$

$2A+2B-C=0........(3)$

$2A-2B-D=0........(4)$

Solving the above equations we get,

$$A=\frac{1}{6},B=-\frac{1}{6},C=0,D=\frac{2}{3}$$

Our Integral becomes,

$$\int \frac{x^2}{(x-1)(x+1)(x^2+2)}dx=\int \frac{1}{6(x-1)}-\frac{1}{6(x+1)}+\frac{2}{3(x^2+2)}dx$$

$$=\frac{1}{6}\log(x-1)-\frac{1}{6}\log(x+1)+\frac{2}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

$$=\frac{1}{6}\left[\log(x-1)-\log(x+1)+2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]+C$$

20. To evaluate: $\int_0^2 \left(x^2 + e^x\right) dx$

Here $f(x) = x^2 + e^x$, $a = 0, b = 2$

So, $nh = b - a = 2$

Now $f(0) = 0 + e^{\circ} = 1$

$f(0 + h) = f(h) = h^2 + e^h$

$f(0 + 2h) = f(2h) = 2^2h^2 + e^{2h}$

$f(0 + (n - 1)h) = f((n - 1)h) = (n - 1)^2h^2 + e^{(n-1)h}$

Now $\int_0^2 f(x)dx = \lim_{h\to 0}\left[f(0) + f(0+h) + f(0+2h) + \ldots + f(0+(n-1)h)\right]$

$$= \lim_{h\to o} h\left[\left(h^2 + 2^2h^2 + \ldots (n-1)^2 h^2\right) + \left(1 + e^h + e^{2h} + \ldots + e^{(n-1)h}\right)\right]$$

$$= \lim_{h\to o} h\left[h^2\left(1^2 + 2^2 + \ldots + (n-1)^2\right) + 1.\left(\frac{\left(e^h\right)^n - 1}{e^h - 1}\right)\right]$$

$$= \lim_{h\to o}\left[h^3 \frac{n(n-1)(2n-1)}{6} + \frac{h\left(e^{nh} - 1\right)}{e^h - 1}\right]$$

$$= \lim_{h\to o}\left[\frac{nh(nh-h)(2nh-h)}{6}\right] + \lim_{h\to o}\left(e^{nh} - 1\right) \times \lim_{h\to o} \frac{h}{e^h - 1}$$

$$= \frac{2(2-0)(4-0)}{6} + \left(e^2 - 1\right) \times 1$$

$$= \frac{8}{3} + e^2 - 1 = \frac{5}{3} + e^2$$

21. The given differential equation is

$$(x+y+1)^2\frac{dy}{dx}=1$$

sub $x+y+1=v$

$$\frac{dy}{dx}+1=\frac{dv}{dx}$$

so,

$$\Rightarrow v^2\left(\frac{dv}{dx}-1\right)=1$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{v^2}+1$$

$$\Rightarrow \frac{dv}{dx}=\frac{1+v^2}{v^2}$$

$$\Rightarrow \int\frac{v^2}{1+v^2}dv=\int dx$$

$$\Rightarrow \int\left(1-\frac{1}{1+v^2}\right)dv=x+c$$

$$\Rightarrow v-\tan^{-1}v=x+c$$

$$\Rightarrow x+y+1-\tan^{-1}(x+y+1)=x+c$$

given that $x=-1$, then $y=0$

we get

$c=1$

so,

$$y=\tan^{-1}(x+y+1)$$

$$\tan y=x+y+1$$

OR

The given differential equation is

$(x-y)(dx+dy)=dx-dy$

$\Rightarrow (x-y-1)dx=-(x-y+1)dy$

$\Rightarrow \frac{dy}{dx}=-\frac{x-y+1}{x-y-1}$

let $x-y=v$

$\frac{dy}{dx}=1-\frac{dv}{dx}$

so,

$1-\frac{dv}{dx}=-\frac{v-1}{v+1}$

$\Rightarrow \frac{dv}{dx}=\frac{2v}{v+1}$

$\Rightarrow \frac{v+1}{v}\,dv=2\,dx$

$\Rightarrow \int\left(1+\frac{1}{v}\right)dv=\int dx$

$\Rightarrow v+\log|v|=2x+c$

$\Rightarrow x-y+\log|x-y|=2x+c$

$\Rightarrow \log|x-y|=x+y+c$

given that

$x=0$, then $y=-1$

so, substituting we get

$c=1$

$\log|x-y|=x+y+1$

$x-y=\pm e^{x+y+1}$

22. $\vec{a}=\hat{i}+4\hat{j}+2\hat{k},\ \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$

The vector which is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is in the direction of

$\vec{a}\times\vec{b}$. $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix}=32\hat{i}-\hat{j}-14\hat{k}$

Since $\vec{d}$ is perpendicular to $\vec{a}$ and $\vec{b}$

$\vec{a}\times\vec{b}$ is also perpendicular to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{d}$ is parallel to $\vec{a}\times\vec{b}$.

$$\Rightarrow \vec{d}=\lambda\left(\vec{a}\times\vec{b}\right)$$

$$\Rightarrow \vec{d}=32\lambda\hat{i}-\lambda\hat{j}-14\lambda\hat{k}$$

Also $\vec{c}.\vec{d}=15 \Rightarrow 2.32\lambda+(-1)(-\lambda)+4(-14\lambda)=15$

$$\Rightarrow \lambda=\frac{5}{3}$$

$$\Rightarrow \vec{d}=\frac{5}{3}\left(32\hat{i}-\hat{j}-14\hat{k}\right)$$

23.

$$\frac{1-x}{3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$$

and

$$\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{6-z}{7}$$

Let us rewrite the equations of the given lines as follows:

$$\frac{-(x-1)}{3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$$

and

$$\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{-(z-6)}{7}$$

That is we have,

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$$

and

$$\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-7}$$

The lines are perpendicular so angle between them is 90°

So, $\cos\theta=0$

Here $(a_1, b_1, c_1)=(-3, 2\lambda, 2)$ and $(a_2, b_2, c_2)=(3\lambda, 1, -7)$

For perpendicular lines

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$\Rightarrow -9\lambda+2\lambda-14=0$$
$$\Rightarrow -7\lambda-14=0$$
$$\Rightarrow -7\lambda=14$$
$$\Rightarrow \lambda=\frac{14}{-7}$$
$$\Rightarrow \lambda=-2$$

SECTION – D

24.

$$A=\begin{bmatrix}1 & -1 & 0\\ 2 & 1 & 3\\ 1 & 2 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 0 & 1 & 1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & 2 & 1\\ -1 & 1 & 2\\ 0 & 3 & 1\end{bmatrix}, B^T=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 1\\ 3 & 3 & 1\end{bmatrix}$$

$$(A+B)^T=$$

$$\left(\begin{bmatrix}1 & -1 & 0\\ 2 & 1 & 3\\ 1 & 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 0 & 1 & 1\end{bmatrix}\right)^T$$

$$=\begin{bmatrix}1+1 & 1 & 0+3\\ 2+2 & 1+1 & 3+3\\ 1+0 & 2+1 & 1+1\end{bmatrix}^T$$

$$=\begin{bmatrix}2 & 1 & 3\\ 4 & 2 & 6\\ 1 & 3 & 2\end{bmatrix}^T$$

$$=\begin{bmatrix}2 & 4 & 1\\ 1 & 2 & 3\\ 3 & 6 & 2\end{bmatrix}$$

$$A^T+B^T=\begin{bmatrix}1 & 2 & 1\\ -1 & 1 & 2\\ 0 & 3 & 1\end{bmatrix}+\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 1\\ 3 & 3 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1+1 & 2+2 & 1+0\\ -1+2 & 1+1 & 2+1\\ 0+3 & 3+3 & 1+1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4 & 1\\ 1 & 2 & 3\\ 3 & 6 & 2\end{bmatrix}$$

$$(A+B)^T=A^T+B^T$$

OR

$$A=\begin{bmatrix}1&1&-1\\2&2&-3\\-1&0&2\end{bmatrix},B=\begin{bmatrix}4&-2&-1\\-1&1&1\\2&-1&0\end{bmatrix}$$

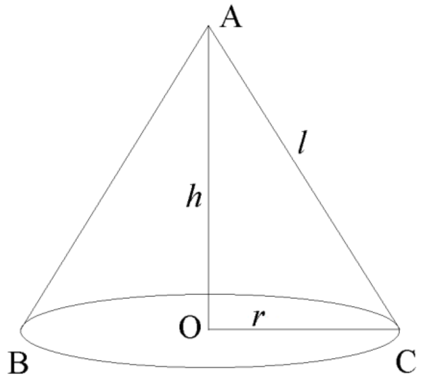
$$AB=\begin{bmatrix}4-1-2&-2+1+1&-1+1\\8-2-6&-4+2+3&-2+2\\-4+4&2-2&1\end{bmatrix}$$

$$AB=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$BA=\begin{bmatrix}4&-2&-1\\-1&1&1\\2&-1&0\end{bmatrix}\begin{bmatrix}1&1&-1\\2&2&-3\\-1&0&2\end{bmatrix}$$

$$BA=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

25.



Here, Volume 'V' of the cone is $V = \frac{1}{3}\pi r^2 h \Rightarrow r^2 = \frac{3V}{\pi h}$ ...(1)

Surface area $S = \pi r l = \pi r\sqrt{h^2 + r^2}$ ...(2)

Where h = height of the cone

r = radius of the cone

l = Slant height of the cone

$S^2 = \pi^2 r^2 \left(h^2 + r^2\right)$ from equation (2)

Let, $S_1 = S^2$

Substituting the value of r^2 from equation (1), we have,

$$S_1 = \frac{3\pi V}{h}\left(h^2 + \frac{3V}{\pi h}\right) = 3\pi V h + \frac{9V^2}{h^2}$$

Differentiating S_1 with respect to h, we get

$$\frac{dS_1}{dh} = 3\pi V + 9V^2\left(\frac{-2}{h^3}\right)$$

$\frac{dS_1}{dh} = 0$ for maxima/minima

$$3\pi V + 9V^2\left(\frac{-2}{h^3}\right) = 0$$

$$\Rightarrow 3\pi V = 9V^2\left(\frac{2}{h^3}\right)$$

$$\Rightarrow h^3 = \frac{6V}{\pi}$$

$$\frac{d^2S_1}{dh^2}=\frac{54V^2}{h^4}$$

$$\frac{d^2S_1}{dh^2}>0 \text{ at } h^3=\frac{6V}{\pi^2}$$

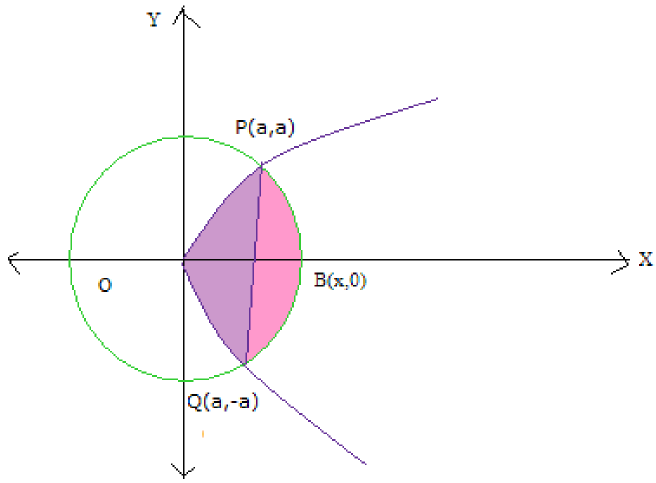
Therefore curved surface area is minimum at $\frac{\pi h^3}{6}=V$

Thus, $\frac{\pi h^3}{6}=\frac{1}{3}\pi r^2 h \;\Rightarrow\; h^2=2r^2$

$\Rightarrow\; h=\sqrt{2}r$

Hence for least curved surface the altitude is $\sqrt{2}$ times radius.

26.



The circle is $x^2 + y^2 = 2a^2 \Rightarrow C(0, \sqrt{2}a)$

The parabola is $y^2 = ax, a > 0 \Rightarrow y^2 = 4\frac{1}{4}ax, a > 0$

Their point of intersection is given by : $x^2 + ax = 2a^2$

$\Rightarrow x^2 + ax - 2a^2 = 0$

$\Rightarrow (x+2a)(x-a) - 0$

$\Rightarrow x = a, -2a$

$\Rightarrow x = a$

$\Rightarrow y^2 = a^2 \Rightarrow y = \pm a$

$\Rightarrow$ shade region is the smaller of the two areas
in which the circle is divided by the parabola

$$A=2\left[\int_0^a \sqrt{ax}dx + \int_a^{\sqrt{2}a} \left(2a^2 - x^2\right)^{1/2} dx\right]$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a + 2\left[\frac{x}{2}\left(2a^2 - x^2\right)^{1/2} + \frac{2a^2}{2}\sin^{-1}\frac{x}{\sqrt{2}a}\right]_a^{\sqrt{2}a}$$

$$= \frac{4}{3}\sqrt{a}\left[a^{3/2}\right] + \left[x\left(2a^2 - x^2\right)^{1/2} + 2a^2\sin^{-1}\frac{x}{\sqrt{2}a}\right]_a^{\sqrt{2}a}$$

$$=\frac{4}{3}\sqrt{a}\left[a^{3/2}\right]+\left[2a^2\sin^{-1}\frac{\sqrt{2}a}{\sqrt{2}a}-a\left(2a^2-a^2\right)^{1/2}-2a^2\sin^{-1}\frac{a}{\sqrt{2}a}\right]$$

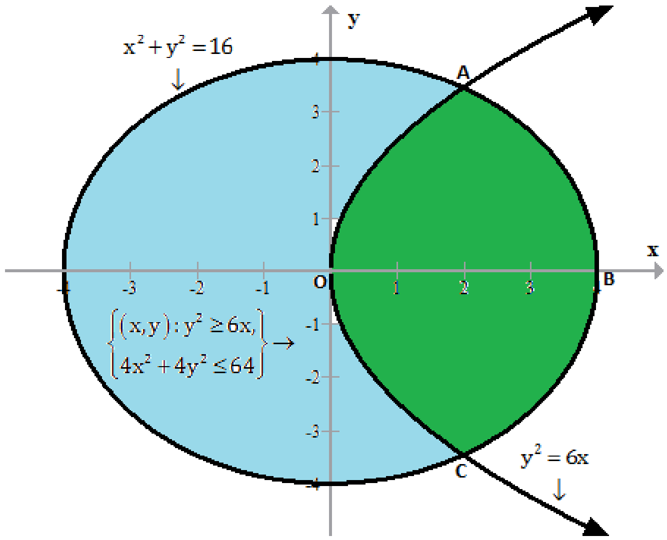
$$=\frac{4}{3}\sqrt{a}\left[a^{3/2}\right]+\left[2a^2\sin^{-1}1-a\left(a^2\right)^{1/2}-2a^2\sin^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=\frac{4}{3}\sqrt{a}\left[a^{3/2}\right]+\left[2a^2\frac{\pi}{2}-a^2-2a^2\frac{\pi}{4}\right]$$

$$=\frac{4}{3}\sqrt{a}\left[a^{3/2}\right]+2a^2\frac{\pi}{4}-a^2$$

$$=\frac{4}{3}\sqrt{a}\left[a^{3/2}\right]+a^2\frac{\pi}{2}-a^2\text{sq. units}$$

OR



$4x^2 + 4y^2 = 64$

$\Rightarrow x^2 + y^2 = 16$

The points of intersection of the two curves $x^2 + y^2 = 16$ and $y^2 = 6x$

$x^2 + 6x = 16 \Rightarrow x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0 \Rightarrow x = -8, 2$

But x is non negative so $x = 2$

Required area (Blue shaded portion)

= Ar (Circle) – Ar (Green Shaded portion)

$$\Rightarrow \text{Required area} = \pi(4)^2 - 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16-x^2}\,dx\right]$$

$$= 16\pi - 2\sqrt{6}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^2 - 2\left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4$$

$$= 16\pi - \frac{4\sqrt{6}}{3}\left(2^{\frac{3}{2}} - 0\right) - 2\left[8\sin^{-1}(1) - \sqrt{16-4} - 8\sin^{-1}\left(\frac{1}{2}\right)\right]$$

$$= 16\pi - \frac{4\sqrt{6} \times 2\sqrt{2}}{3} - 2\left[8 \times \frac{\pi}{2} - \sqrt{12} - 8 \times \frac{\pi}{6}\right]$$

$$= 16\pi - \frac{16\sqrt{3}}{3} - \frac{16\pi}{3} + 4\sqrt{3} = \frac{32\pi}{3} - \frac{4\sqrt{3}}{3} = \frac{4}{3}\left(8\pi - \sqrt{3}\right) \text{sq.units}$$

27. Let $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\lambda$

$x=-2+3\lambda, y=-1+2\lambda, z=3+2\lambda$

Therefore, a point on this line is: {(-2+3λ), (-1 + 2λ), (3 + 2λ)}

The distance of the point {(-2+3λ), (-1 + 2λ), (3 + 2λ)} from point (1, 2, 3) = $3\sqrt{2}$

$\therefore \sqrt{(-2+3\lambda-1)^2+(-1+2\lambda-2)^2+(3+2\lambda-3)^2}=3\sqrt{2}$

$\Rightarrow (-3+3\lambda)^2+(-3+2\lambda)^2+(2\lambda)^2=18$

$\Rightarrow 9+9\lambda^2-18\lambda+9+4\lambda^2-12\lambda+4\lambda^2=18$

$17\lambda^2-30\lambda=0$

$\lambda=0, \lambda=\frac{30}{17}$

When $\lambda=\frac{30}{17}$,

$x=-2+3\lambda=-2+3\left(\frac{30}{17}\right)=-2+\frac{90}{17}=\frac{56}{17}$

$y=-1+2\lambda=-1+2\left(\frac{30}{17}\right)=-1+\frac{60}{17}=\frac{43}{17}$

$z=3+2\lambda=3+2\left(\frac{30}{17}\right)=\frac{51+60}{17}=\frac{111}{17}$

Thus, when $\lambda=\frac{30}{17}$, the point is $\left(\frac{56}{17},\frac{43}{17},\frac{111}{17}\right)$

and when λ = 0, the point is (-2, -1, 3).

OR

The equation of line L_1 :

$$\frac{1-x}{3}=\frac{7y-14}{p}=\frac{z-3}{2}$$

$$\Rightarrow \frac{x-1}{-3}=\frac{y-2}{\frac{p}{7}}=\frac{z-3}{2}....(1)$$

The equation of line L_2 :

$$\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$$

$$\Rightarrow \frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}....(2)$$

Since line L_1 and L_2 are perpendicular to each other, we have

$$-3\times\left(\frac{-3p}{7}\right)+\frac{p}{7}\times1+2\times(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{p}{7}=10$$

$$\Rightarrow 10p=70$$

$$\Rightarrow p=7$$

Thus equations of lines L_1 and L_2 are:

$$\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-3}{2}$$

$$\frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Thus the equation of the line passing through the point $(3, 2, -4)$ and parallel to the line L_1 is:

$$\frac{x-3}{-3}=\frac{y-2}{1}=\frac{z+4}{2}$$

28. Let x and y be number of packets of food P and Q respectively.

Linear programming problem is

Minimize Z = 6x + 3y (Vitamins A)

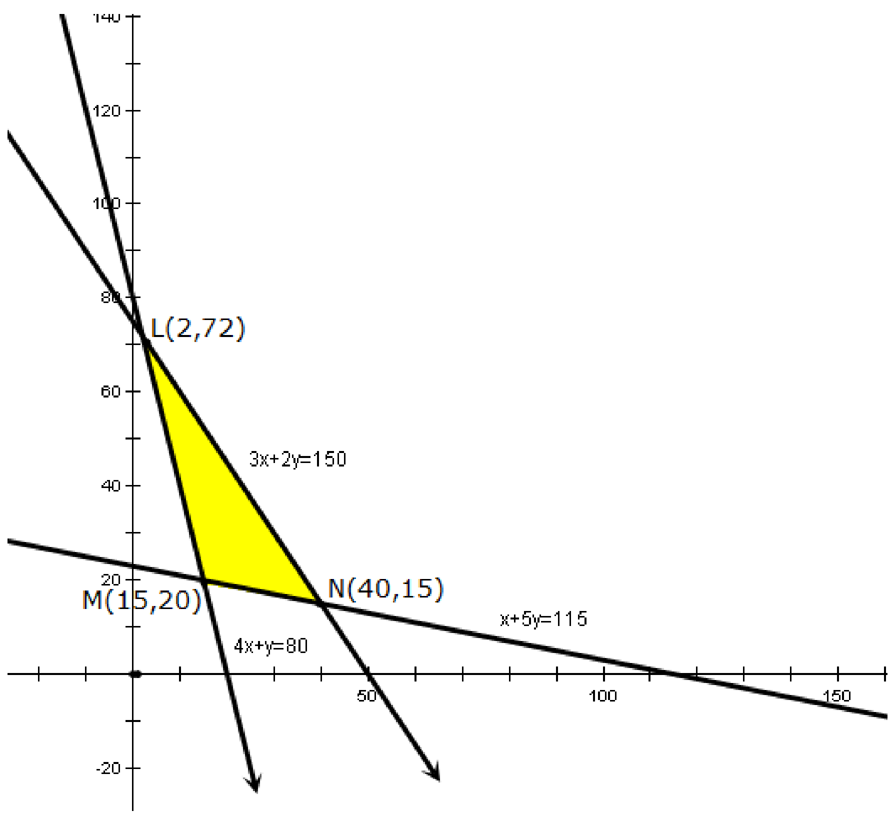
s.t. $12x + 3y \geq 240$ or $4x + y \geq 80$

$4x + 20y \geq 460$ or $x + 5y \geq 115$

$6x + 4y \leq 300$ or $3x + 2y \leq 150$

$x \geq 0, y \geq 0,$

Graphically the problem can be represented as



Co-ordinates of corner points L, M, N are (2, 72), (15, 20) and (40, 15), we have

Corner points	z = 6x + 3y
L (2, 72)	228
M (15, 20)	150 → Minimum
N (40, 15)	285

Hence, minimum vitamin A is used at point (15, 20) i.e. when 15 packets of food P and 20 packets of food Q are used.

29. This is a case of Bernoulli's trials.

Let Success: Getting a purple ball on a draw

Failure: Getting a pink ball on a draw

$$p = P(\text{success}) = \frac{10}{25} = \frac{2}{5} \Rightarrow q = \frac{3}{5}$$

$$\text{(i)}P(6\text{success}) = {}^6C_6 p^6 q^0 = 1 \times \left(\frac{2}{5}\right)^6 \times 1 = \left(\frac{2}{5}\right)^6$$

(ii)P(not more than 2 failures) = P(not less than 4 success) = P(4) + P(5) + P(6)

$$= {}^6C_4 p^4 q^2 + {}^6C_5 p^5 q^1 + {}^6C_6 p^6 q^0 = 15 \times \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^2 + 6 \times \left(\frac{2}{5}\right)^5 \left(\frac{3}{5}\right) + 1 \times \left(\frac{2}{5}\right)^6$$

$$= \left(\frac{2}{5}\right)^4 \left[15 \times \left(\frac{3}{5}\right)^2 + 6 \times \left(\frac{2}{5}\right)\left(\frac{3}{5}\right) + \left(\frac{2}{5}\right)^2\right]$$

$$= \left(\frac{2}{5}\right)^4 \left[\frac{135}{25} + \frac{36}{25} + \frac{4}{25}\right]$$

$$= \left(\frac{2}{5}\right)^4 \left[\frac{175}{25}\right] = 7 \times \left(\frac{2}{5}\right)^4$$

$$\text{(iii)}P(3\text{success } 3\text{failures}) = P(3) = {}^6C_3 p^3 q^3 = 15\left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^3 = \frac{648}{3125}$$

(iv)P(atleast 1 failure)=P(at most 5 success)

$$= P(0) + P(1) + P(2) + P(3) + P(4) + P(5) = 1 - P(6) = 1 - {}^6C_6 p^6 q^0$$

$$= 1 - 1 \times \left(\frac{2}{5}\right)^6 \times 1 = 1 - \left(\frac{2}{5}\right)^6 = 1 - \frac{64}{15625} = \frac{15561}{15625}$$