MEGHALAYA
Class XII
Mathematics
SAMPLE PAPER-2

Time allowed: 3 hours **Maximum Marks: 100**

Section A

1. a) Let $I = \int_0^p \frac{\sqrt{x}}{\sqrt{x}+\sqrt{p-x}}dx$ ...(1)

According to property,

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^p \frac{\sqrt{p-x}}{\sqrt{p-x}+\sqrt{x}}dx \text{ ...(2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^p \frac{\sqrt{x}+\sqrt{p-x}}{\sqrt{x}+\sqrt{p-x}}dx$$

$$= \int_0^p 1dx = [x]_0^p = p - 0 = p$$

Thus, $2I = p \Rightarrow I = \frac{p}{2}$

b) * is a binary operation on R – {0} defined by $a * b = \frac{2ab}{3}$

If e is an identity then a*e = a = e*a

$$\frac{2ae}{3} = a \Rightarrow e = \frac{3}{2}$$

2. a)

$$\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

$C_3 \to C_3 + C_2$

$$\Rightarrow \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$$

$$\Rightarrow (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0$$

b)

$$\text{Let } A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

Transpose of $(A - A') = (A - A')'$

$$= \begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix} = -(A - A')$$

$\Rightarrow (A - A')$ is a skew symmetric matrix

3. a)

$y = a\sin(x+b)$ ...(1)

Differentiating w.r.t. x,

$$\frac{dy}{dx} = a\cos(x+b)$$

Again differentiating w.r.t. x we get ,

$$\frac{d^2y}{dx^2} = -a\sin(x+b)$$

$$= -y$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = 0$$

This is the required. D. E.

b)

$$\text{Let } \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\alpha, \text{so that } \alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\Rightarrow \sin\alpha=-\frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin\alpha=-\sin\frac{\pi}{3}$$

$$\Rightarrow \sin\alpha=\sin\left(-\frac{\pi}{3}\right)$$

$$\text{We know that } -\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\therefore \alpha=-\frac{\pi}{3}$$

4. a)

$$\cos\left(\sin^{-1}\frac{1}{3}+\cos^{-1}x\right)=0$$

$$\Rightarrow \sin^{-1}\frac{1}{3}+\cos^{-1}x=\cos^{-1}0=\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\frac{1}{3}+\cos^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow x=\frac{1}{3}$$

b) Given vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=-\hat{i}+2\hat{j}-3\hat{k}$

If $\vec{a}\perp\vec{b}$

2 x (-1) + λ (2) + (1)(-3) = 0

$\Rightarrow 2\lambda = 5 \Rightarrow \lambda = \frac{5}{2}$

5. $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$

$$\Rightarrow \vec{x}.\vec{x}-\vec{a}.\vec{a}=15$$

$$\Rightarrow |\vec{x}|^2-|\vec{a}|^2=15$$

$$\Rightarrow |\vec{x}|^2-1=15$$

$$\Rightarrow |\vec{x}|^2=16$$

$$\Rightarrow |\vec{x}|=\pm 4$$

6. $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k};$

$\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\Rightarrow \ \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$

$\dfrac{\vec{a} + \vec{b}}{2} = 3\hat{i} + \hat{j} - 4\hat{k}$

7.

$a_{ij} = \dfrac{|2i - j|}{3j}$

$\Rightarrow a_{11} = \dfrac{|2 \times 1 - 1|}{3 \times 1} = \dfrac{1}{3}; a_{12} = \dfrac{|2 \times 1 - 2|}{3 \times 2} = 0$

$a_{21} = \dfrac{|2 \times 2 - 1|}{3 \times 1} = \dfrac{3}{3} = 1; a_{22} = \dfrac{|2 \times 2 - 2|}{3 \times 2} = \dfrac{2}{6} = \dfrac{1}{3}$

$\therefore$ The required matrix is $\begin{bmatrix} \frac{1}{3} & 0 \\ 1 & \frac{1}{3} \end{bmatrix}$

8. $\dfrac{d\left(e^{x^2 + \tan x}\right)}{dx} = e^{x^2 + \tan x} \dfrac{d}{dx}(x^2 + \tan x) = e^{x^2 + \tan x}\left(2x + \sec^2 x\right)$

Differentiating w.r.t x

$\dfrac{dy}{dx} = -\dfrac{a}{x^2}$ ....(i)

$\dfrac{d^2y}{dx^2} = \dfrac{2a}{x^3}$ ....(ii)

Since $\dfrac{2a}{x^3} = \dfrac{2}{x}\left(\dfrac{a}{x^2}\right) \Rightarrow \dfrac{2a}{x^3} - \dfrac{2}{x}\left(\dfrac{a}{x^2}\right) = 0$

$\Rightarrow \dfrac{2a}{x^3} + \dfrac{2}{x}\left(-\dfrac{a}{x^2}\right) = 0$

using (i) and (ii)

$\Rightarrow \dfrac{d^2y}{d^2x} + \dfrac{2}{x}\left(\dfrac{dy}{dx}\right) = 0$

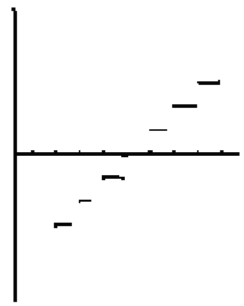
$\Rightarrow y = \dfrac{a}{x} + b$ is a solution of $\dfrac{d^2y}{d^2x} + \dfrac{2}{x}\left(\dfrac{dy}{dx}\right) = 0$

9. $\sin\left[\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right] = \sin\left[\dfrac{\pi}{3} - \left(-\dfrac{\pi}{6}\right)\right]$

$= \sin\left[\dfrac{\pi}{3} + \dfrac{\pi}{6}\right] = \sin\dfrac{\pi}{2} = 1$

10. If a vertical line intersects the graph of a relation in two or more points, then the relation is *not* a function.

Graph should have no vertical lines



11.

A = {1, 2, 3, 4, 5}

Now 1 * 2 = L.C.M. (1, 2) = 2

1 * 3 = 3, 1 * 4 = 4, 1 * 5 = 5

2 * 3 = 6 $\notin$ A

So * is not a binary operation on A

12. Equation of a line passing through (a, b, c) and with direction cosines (l, m, n) is given by

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

Here point is (0, 0, 0)

Direction cosines of the line parallel to the x-axis are 1, 0, 0

$\therefore$ required line is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

13. If A = { a, b, c } and B = { 1, 2, 3 } and a function

f : A $\to$ B is given by f = {(a, 2) , (b , 3) , (c , 1)}

Every element of set A is mapped to an unique element of set B, i.e. each element in set B has an unique pre image in B.

$\Rightarrow$ f is a one–one function

Range of f = {1, 2, 3} = B

$\Rightarrow$ f is an onto function

$\therefore$ f is a bijective function

14.

$$I=\int \frac{1}{\sqrt{9-25x^2}}dx$$

$$=\frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}dx$$

$$=\frac{1}{5}\times\frac{5}{3}\sin^{-1}\left(\frac{x}{\frac{3}{5}}\right)+c$$

$$=\frac{1}{3}\sin^{-1}\left(\frac{5x}{3}\right)+c$$

15.

$P(-1,-2,4)$ and $Q(2,0,-2)$

Position vector of P = $-1\hat{i}-2\hat{j}+4k$

Position vector of Q = $2\hat{i}+0\hat{j}-2k$

$\overrightarrow{PQ}=$ Position vector of Q $-$ Position vector of P

$=\left(2\hat{i}+0\hat{j}-2k\right)-\left(-1\hat{i}-2\hat{j}+4k\right)$

$=3\hat{i}+2\hat{j}-6k$

Magnitude of $\overrightarrow{PQ}=\sqrt{3^2+2^2+(-6)^2}=\sqrt{49}=7$

Section B

16. Let $x = \sin\alpha$ and $y=\sin\beta$, such that $\alpha,\beta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Substituting the values of x and y in the equation

$\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)$, we have,

$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$

$\therefore \cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=2a\left(\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right)$$

$$\cos\frac{\alpha-\beta}{2}=a\left(\sin\frac{\alpha-\beta}{2}\right)$$

$$\cot\frac{\alpha-\beta}{2}=a$$

$$\frac{\alpha-\beta}{2}=\cot^{-1}a$$

$\alpha-\beta=2\cot^{-1}a$

$\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

Differentaiting with respect to x, we have

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\times\frac{dy}{dx}=0$$

$$\Rightarrow\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

17. Here, $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

$\Rightarrow \left(\vec{a}+\vec{b}\right)^2=\left(-\vec{c}\right)^2$

$\Rightarrow \vec{a}^2+\vec{b}^2+2\vec{a}.\vec{b}=\vec{c}^2$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta=|\vec{c}|^2$,

where θ is the angle between $\vec{a}$ and $\vec{b}$

and which is perpendicular to the plane

$$\Rightarrow \theta=\cos^{-1}\left(\frac{1}{2}\right)=60^o$$

OR

$\vec{a} = \hat{i} - \lambda\hat{j} + 3k$ and $\vec{b} = 4\hat{i} - 5\hat{j} + 2k$

vectors are perpendicular if $\vec{a} \cdot \vec{b} = 0$

$\vec{a} \cdot \vec{b} = \left(\hat{i} - \lambda\hat{j} + 3k\right) \cdot \left(4\hat{i} - 5\hat{j} + 2k\right)$

$= 1 \times 4 + (-\lambda) \times (-5) + 3 \times 2 = 4 + 5\lambda + 6 = 10 + 5\lambda$

$\vec{a} \cdot \vec{b} = 0 \Rightarrow 10 + 5\lambda = 0 \Rightarrow \lambda = -2$

18. Given Integral = $\int x^{2x}(1 + \log x)\, dx = \int x^x . x^x (1 + \log x) dx$

Let $x^x = t$

$x^x (1 + \log x).\, dx = dt$

$\Rightarrow I = \int t.dt$

$$= \frac{t^2}{2} + C = \frac{1}{2}\left(x^x\right)^2 + C = \frac{x^{2x}}{2} + C$$

19.

X	0	1	2	3	4	5
P(X)	0.1	K	0.2	2K	0.3	K

$$\sum_{i=1}^{n} P(X = x_i) = 1$$

(i) Since

So 0.1 + K + 0.2 +2K + 0.3 + K = 1

$\Rightarrow$ 4K = 0.4 $\Rightarrow$K = 0.1

(ii) P(X$\leq$ 1) = P(0) + P(1) = 0.1 + 0. 1 = 0.2

P(X > 3)= P(4) + P(5) = 0.3 + 0.1 = 0.4

20.

$$I = \int \frac{x}{\sqrt{8+x-x^2}}dx$$

$$\text{Let } x = A\left[\frac{d}{dx}(8+x-x^2)\right] + B$$

$$x = A(1 - 2x) + B$$

$$\Rightarrow x = -2Ax + (A+B)$$

$$\Rightarrow -2A = 1; A + B = 0$$

$$\Rightarrow A = -\frac{1}{2}; B = \frac{1}{2}$$

$$\therefore I = \int \frac{-\frac{1}{2}(1-2x)+\frac{1}{2}}{\sqrt{8+x-x^2}}dx$$

$$\Rightarrow I = -\frac{1}{2}\int \frac{(1-2x)}{\sqrt{8+x-x^2}}dx + \frac{1}{2}\int \frac{1}{\sqrt{8+x-x^2}}dx = I_1 + I_2 (\text{say})$$

$$I_1 = -\frac{1}{2}\int \frac{(1-2x)}{\sqrt{8+x-x^2}}dx;$$

$$\text{Let } t = 8+x-x^2 \therefore dt = (1-2x)dx$$

$$\Rightarrow I_1 = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\left[2\sqrt{t}\right] = -\sqrt{8+x-x^2}$$

$$I_2 = \frac{1}{2}\int \frac{1}{\sqrt{8+x-x^2}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{33}{4}-\left(x-\frac{1}{2}\right)^2}} = \frac{1}{2}\sin^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{33}}{2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right)$$

$$\text{So } I = -\sqrt{8+x-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x-1}{\sqrt{33}}\right) + C$$

OR

$\text{Let } I = \int \frac{1}{3+\sin^2 x} dx$

Dividing both numerator and denominator by $\cos^2 x$

$= \int \frac{\sec^2 x}{3\sec^2 x + \tan^2 x} dx$

$= \int \frac{\sec^2 x}{3(1+\tan^2 x)+\tan^2 x} dx$

$= \int \frac{\sec^2 x}{3+(2\tan x)^2} dx$

Let $2\tan x = t \Rightarrow 2\sec^2 x\, dx = dt$

$\Rightarrow I = \frac{1}{2}\int \frac{1}{(\sqrt{3})^2 + t^2} dt$

$= \frac{1}{2} \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{3}}\right) + c$

Section C

21. Let the two tailors work for x days and y days respectively, The problem is to minimise the objective function, C = 150 x + 200 y, subject to the constraints,

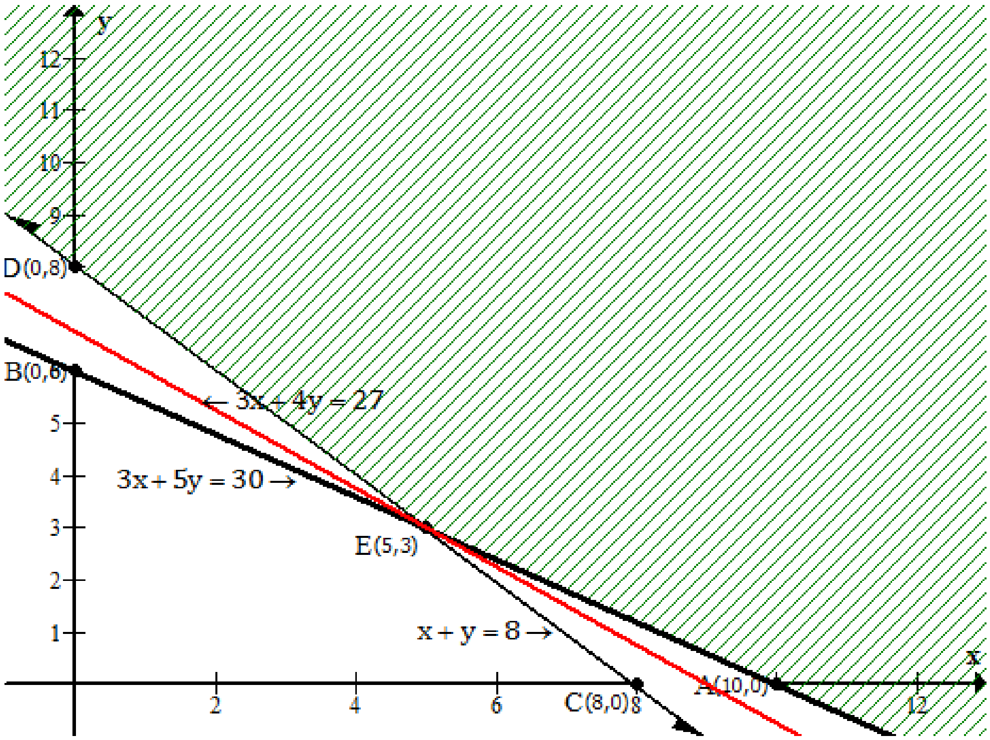
$6x + 10y \geq 60 \Leftrightarrow 3x + 5y \geq 30$

$4x + 4y \geq 32 \Leftrightarrow x + y \geq 8$

And,

$x \geq 0, y \geq 0$

Feasible region is shown shaded.

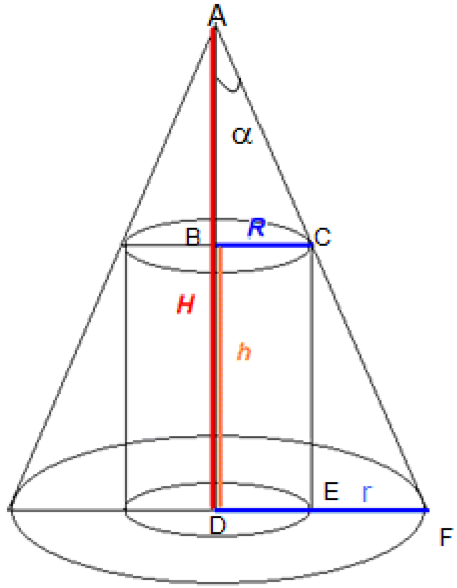


This region is unbounded.

Corner points	Objective function values C =150x + 200y
A(10, 0)	1500
E(5, 3)	1350
D(0, 8)	1600

The red line in the graph shows the line 150 x + 200y = 1350 or 3x + 4y =27
We see that the region 3x + 4y > 27 has no point in common with the feasible region.
Hence, the function has minimum value at E(5, 3).
Hence, the labour cost is the least when tailor A works for 5 days and Tailor B works for 3 days.

22. Let dimensions of the cone be given as



h = height of cone (AD)

r = radius of cone (DF)

Let dimensions of cylinder be given as

H = Height of cylinder (BD)

R = Radius of cylinder (BC)

We have ΔABC ~ ΔADF

$$\Rightarrow \frac{AB}{AD} = \frac{BC}{DF}$$

$$\Rightarrow \frac{h-H}{h} = \frac{R}{r}$$

$$\Rightarrow H = \left(1 - \frac{R}{r}\right)h$$

$$\Rightarrow V = \text{Volume of cylinder} = \pi R^2 H$$

$$\Rightarrow V \quad = \pi R^2\left(h - \frac{Rh}{r}\right)$$

$$= \pi R^2 h - \frac{\pi R^3 h}{r}$$

$$\Rightarrow \frac{dV}{dR} \quad = 2\pi Rh - \frac{3\pi R^2 h}{r}$$

$$\Rightarrow \frac{dV}{dR} = 0 \qquad \Rightarrow \qquad 2\pi Rh = \frac{3\pi R^2 h}{r}$$

$$\Rightarrow \frac{R}{r} = \frac{2}{3} \quad \text{or} \quad R = \frac{2r}{3}$$

$$\frac{d^2V}{dR^2} = 2\pi h - 6\pi \frac{Rh}{r}$$

$$\left.\frac{d^2V}{dR^2}\right]_{\frac{R}{r}=\frac{2}{3}} = 2\pi h - 6\pi h\left(\frac{2}{3}\right) = -2\pi h < 0$$

So maximum volume is when $R = \frac{2r}{3}$ or $\frac{R}{r} = \frac{2}{3}$

$$\text{so, } H = \left(1 - \frac{R}{r}\right)h = \left(1 - \frac{2}{3}\right)h = \frac{h}{3}$$

$$\text{and} \quad \tan\alpha = \frac{r}{h}$$

$$\Rightarrow r = h\tan\alpha$$

$$\Rightarrow R = \frac{2r}{3} = \frac{2h}{3}\tan\alpha$$

So maximum volume $= \pi R^2 H$

$$= \pi\left(\frac{2h}{3}\tan\alpha\right)^2 . \frac{h}{3}$$

$$= \frac{4}{27}\pi h^3 \tan^2\alpha$$

23.

Since $A^{-1} = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $R_3 \to \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Applying $R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$

24. Let E_1 be the event that a red ball is transferred from bag A to bag B

Let E_2 be the event that a black ball is transferred from bag A to bag B

$\therefore E_1$ and E_2 are mutually exclusive and exhaustive.

$P(E_1) = 3/7$; $P(E_2) = 4/7$

Let E be the event that a red ball is drawn from bag B

$$P(E|E_1)=\frac{4+1}{(4+1)+5}=\frac{5}{10}=\frac{1}{2}$$

$$P(E|E_2)=\frac{3+1}{(5+1)+4}=\frac{4}{10}=\frac{2}{5}$$

$$\therefore \text{ Required probability } = P(E_2|E)=\frac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)}$$

$$=\frac{\frac{4}{10}\times\frac{4}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{4}{10}\times\frac{4}{7}}=\frac{\frac{16}{70}}{\frac{3}{14}+\frac{16}{70}}=\frac{\frac{16}{70}}{\frac{31}{70}}=\frac{16}{31}$$

$$\therefore \text{ Required probability } = P(E_1|E)=\frac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{3}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{4}{10}\times\frac{4}{7}}=\frac{\frac{3}{14}}{\frac{3}{14}+\frac{16}{70}}=\frac{\frac{3}{14}}{\frac{31}{70}}=\frac{15}{31}$$

25.

$$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0\Rightarrow x+2y+3z-4=0\text{...(i)}$$

$$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0\Rightarrow 2x+y-z+5=0\text{...(ii)}$$

$$\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0\Rightarrow 5x+3y-6z+8=0\text{....(iii)}$$

A plane P contains the line of intersection of the planes (i) and (ii)

$\therefore$ the equation of P.

$$(x+2y+3z-4)+\lambda(2x+y-z+5)=0$$

$$\Rightarrow (1+2\lambda)x+(2+\lambda)y+(3-\lambda)z+(-4+5\lambda)=0\text{...(iv)}$$

It is perpendicular to a plane(iii)

the equation of P

$$(1+2\lambda)5+(2+\lambda)3+(3-\lambda)(-6)=0$$

$$\Rightarrow 19\lambda=7\Rightarrow\lambda=\frac{7}{19}$$

Putting in (iv)

$$\Rightarrow\left(1+2\left(\frac{7}{19}\right)\right)x+\left(2+\left(\frac{7}{19}\right)\right)y+\left(3-\left(\frac{7}{19}\right)\right)z+\left(-4+5\left(\frac{7}{19}\right)\right)=0.$$

$$\Rightarrow 33x+45y+50z-41=0$$

Section D

26.a) False.

b)False

c) False

d)True

27. a)

$$\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = \begin{vmatrix} \sin(90-80)^\circ & -\cos(90-80)^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix}$$

$$= \begin{vmatrix} \cos 80^\circ & -\sin 80^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = \left(\cos^2 80^\circ - (-\sin^2 80^\circ)\right)$$

$$= \left(\cos^2 80^\circ + \sin^2 80^\circ\right) = 1$$

b)

$$\int \frac{\left(\tan^{-1}x\right)^8}{1+x^2}dx$$

$$\text{Let } \tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \int t^8 dt = \frac{t^9}{9} + c = \frac{\left(\tan^{-1}x\right)^9}{9} + c$$

c) When vectors $(2\hat{i}-3\hat{j})$ and $(\lambda\hat{i}-6\hat{j})$ are parallel to each other, then $(2\hat{i}-3\hat{j}) = k(\lambda\hat{i}-6\hat{j})$

So 6k = 3 or k = $\frac{1}{2}$ and 2 = λk i.e. λ= 4

d) Magnitude of $\alpha\left(\hat{i}+\hat{j}+\hat{k}\right)=1$, for it to be a unit vector.

$$\left|\alpha.(\hat{i}+\hat{j}+\hat{k})\right| = \sqrt{\alpha^2+\alpha^2+\alpha^2} = \sqrt{3\alpha^2} = \alpha\sqrt{3} = 1$$

$$\Rightarrow \alpha = \pm\frac{1}{\sqrt{3}}$$

e)

28. a) Let I = $\int \frac{1+\cot x}{x+\log \sin x}dx$

Put x + log sin x = t

⇒(1 + cot x) dx = dt

so integral I becomes

$$\int \frac{dt}{t} = \log|t| + C$$

$$= \log|x+\log\sin x| + C$$

b)

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \text{then } |A| = a^3$$

Since, a is a nonzero real number so $|A| \neq 0$

If A is an invertible matrix of order n, then $|adj(A)| = |A|^{n-1}$

So $|adj\ A| = |A|^2 = (a^3)^2 = a^6$

c)

$$\int e^{\log \sin x}.\cos x\ dx = \int \sin x.\cos x\ dx$$

$$= \int t.dt = \frac{t^2}{2} + C$$

$$= \frac{\sin^2 x}{2} + C$$

d) Projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a}.\frac{\vec{b}}{|\vec{b}|}$

$$= \left(\hat{i} - 3\hat{k}\right).\frac{\left(3\hat{i} + \hat{j} - 4\hat{k}\right)}{\sqrt{9+1+16}} = \left|\frac{3+12}{\sqrt{26}}\right| = \frac{15}{\sqrt{26}}$$

29. a) Correct option: (iii)

If the order of matrix A is m × n and the order of matrix B is n × p,

Then the order of product matrix C, obtained is m × p.

Here, order of A is 1 × 3 and order of B is 3 × 4.

Hence, order of product matrix will be 1 × 4.

b) Correct option: (i)

As the horizontal line test would tell us. Every value of e^x has a unique pre image in R

So e^x is a one-one function on R

Range of f = $(0, \infty)$

Hence, it is not an onto function, it is an into function.

c) Correct option: (iii)

If a matrix is of order m × n, it has mn elements. Thus, to find all possible orders of a matrix with 6 elements, first we find all ordered pairs of natural numbers, whose product is 6.

Thus all possible ordered pairs are (1, 6), (6, 1), (3, 2), (2, 3).

Hence, the number of possible orders are 4.

d) Correct option: (ii)

By definition $f : X \to Y$ is surjective if and only if for each $y \in Y$, there exist an element $x \in X$ such that $f(x) = y$.

30. a)

$$\int_{-\pi}^{\pi} \sin x dx$$

Consider, $f(x) = \sin x$

$\Rightarrow f(-x) = \sin(-x)$

$\Rightarrow f(-x) = -\sin x$

$\Rightarrow f(-x) = -f(x)$

Function is odd.

$$\Rightarrow \int_{-\pi}^{\pi} \sin x dx = 0$$

b) $\frac{d \operatorname{cosec}^{-1} x}{dx} = \frac{-1}{x\sqrt{x^2 - 1}}$

c)

$$\lim_{x \to 0} \frac{\sin\left(x^2 - x\right)}{x}$$

$$= \lim_{x \to 0} \frac{\sin\left[x(x-1)\right]}{x(x-1)} \times (x-1)$$

$= 1 \times (0 - 1)$

$= -1$

d) $\int 5^{2x} dx = \frac{5^{2x}}{\log 5} \times 2 + c$

31. The equation of the family of circles touching x-axis at the origin is

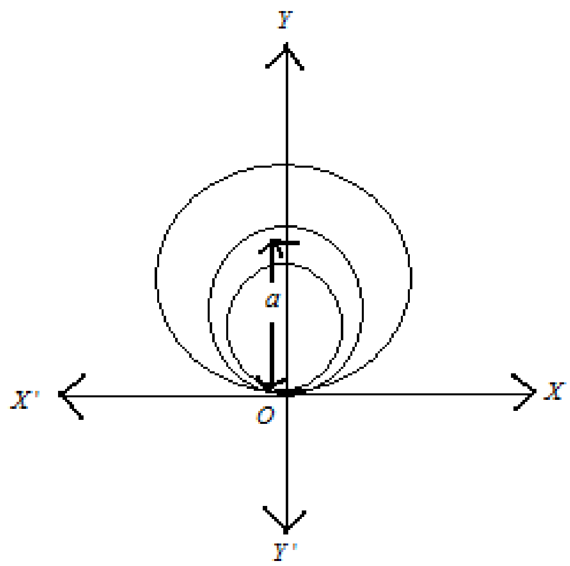
$(x-0)^2 + (y-a)^2 = a^2$, where a is a parameter.

$x^2 + y^2 - 2ay = 0$ ...(i)

This equation contains only one arbitrary constant, thus we ifferentiate it only once, we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow a\frac{dy}{dx} = x + y\frac{dy}{dx}$$



$$\Rightarrow a = \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}$$

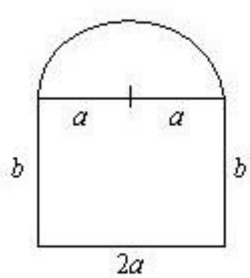
Substituting the value of 'a' in equation (i), we get

$$\Rightarrow x^2 + y^2 = 2y\left\{\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx}}\right\}$$

$$\Rightarrow \left(x^2 + y^2\right)\frac{dy}{dx} = 2y\left(x + y\frac{dy}{dx}\right)$$

This is the required differential equation of all the circles touching the x-axis at the origin.

32. Let 2a cm be the length and b cm be the breadth of the rectangle. Then a cm is the radius of the semi-circle.



By hypothesis, perimeter

$$p = 2a + b + b + \pi a$$

$$\Rightarrow \quad 2b = p - \left(\pi + 2\right)a \qquad .....(1)$$

Also A = Area of the window

$$=\frac{1}{2}\pi a^2+2a\times b$$

$$=\frac{1}{2}\pi a^2+a\times\left[p-(\pi+2)a\right] \qquad [\text{By}(1)]$$

$$=pa-\frac{1}{2}(\pi+4)a^2$$

$$\Rightarrow\frac{dA}{da}=p-(\pi+4)a$$

For max. or min.,

$$\frac{dA}{da}=0$$

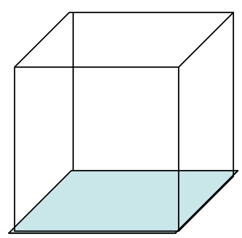
$$\Rightarrow \quad p-(\pi+4)\,a=0$$

$$\Rightarrow \quad a=\frac{p}{\pi+4}.$$

Also $\frac{d^2A}{da^2}=-(\pi+4)<0.$

$\therefore$ The light will be maximum when the radius of the semi-circle is, $a=\frac{p}{\pi+4}$.

OR



Let x be side of the square base and y be the height of the cuboid

Volume (v) = x. x. y = x^2 y

Surface area (S) = 2 (x.x + x.y + x.y) = $2x^2 + 4xy = 2x^2 + 4x\frac{V}{x^2}$

$S = 2x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 4x - \frac{4V}{x^2}$

For minimum S, $\frac{dS}{dx} = 0 \Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = V \Rightarrow x = \sqrt[3]{V}$

$$\frac{d^2S}{dx^2}=4+\frac{8V}{x^3}\Rightarrow\left.\frac{d^2S}{dx^2}\right]_{x=\sqrt[3]{V}}=4+\frac{8V}{V}>0$$

$\therefore$ For $x=\sqrt[3]{V}$, surface area is minimum

$\Rightarrow x^3=V\Rightarrow x^3=x^2y\left[\text{From (i)}\right]\Rightarrow x=y\Rightarrow$ cuboid is a cube

33.

The given line is $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$

Given point is $(2, 4, -1)$

The distance of a point whose position vector is $\overrightarrow{a_2}$

from a line whose vector equation is $\vec{r}=\vec{a_1}+\lambda\vec{v}$ is

$$d=\frac{\left|\vec{v}\times(\vec{a_2}-\vec{a_1})\right|}{\left|\vec{v}\right|}$$

$$=\frac{\left|\left(\hat{i}+4\hat{j}-9k\right)\times\left((2\hat{i}+4\hat{j}-1k)-(-5\hat{i}-3\hat{j}+6k)\right)\right|}{\left|\left(\hat{i}+4\hat{j}-9k\right)\right|}$$

$$=\frac{\left|\left(\hat{i}+4\hat{j}-9k\right)\times\left((2\hat{i}+4\hat{j}-1k)-(-5\hat{i}-3\hat{j}+6k)\right)\right|}{\left|\left(\hat{i}+4\hat{j}-9k\right)\right|}$$

$$=\frac{\left|\left(\hat{i}+4\hat{j}-9k\right)\times(7\hat{i}+7\hat{j}-7k)\right|}{\left|\left(\hat{i}+4\hat{j}-9k\right)\right|}$$

$$\left(\hat{i}+4\hat{j}-9k\right)\times(7\hat{i}+7\hat{j}-7k)=\begin{vmatrix}\hat{i} & \hat{j} & k\\ 1 & 4 & -9\\ 7 & 7 & -7\end{vmatrix}=35\hat{i}-56\hat{j}-21k$$

$$=\frac{7}{\sqrt{98}}\left|\left(5\hat{i}-8\hat{j}-3k\right)\right|=\frac{7}{\sqrt{98}}\sqrt{98}=7\ \text{units}$$

34. $(x-1)dy+y\,dx = x(x-1)\,y^{\frac{1}{3}}\,dx$

$$\Rightarrow (x-1)\frac{dy}{dx}+y = x(x-1)\,y^{\frac{1}{3}}$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{(x-1)} = xy^{\frac{1}{3}}$$

$$\Rightarrow \frac{1}{y^{\frac{1}{3}}}\frac{dy}{dx}+\frac{y^{\frac{2}{3}}}{x-1}=x$$

Let $y^{2/3} = t$

$$\Rightarrow \frac{2}{3}y^{-1/3}\frac{dy}{dx}=\frac{dt}{dx}$$

$$\Rightarrow \frac{3}{2}\frac{dt}{dx}+\frac{t}{x\text{-}1}=x$$

$$\Rightarrow \frac{dt}{dx}+\frac{2}{3}\frac{t}{x\text{-}1}=\frac{2}{3}x$$

This is a linear differential equation, of the form $\frac{dt}{dx}+Pt=Q,$

$$P=\frac{2}{3}\left(\frac{1}{x-1}\right),Q=\frac{2}{3}x$$

Therefore, integrating factor is

$$IF=e^{\int\frac{2}{3}\left(\frac{1}{x-1}\right)dx}=e^{\frac{2}{3}\log(x-1)}=e^{\log(x-1)^{2/3}}=(x-1)^{\frac{2}{3}}$$

$\therefore$ Solution of differential equation is

$$t(x-1)^{2/3}=\int\frac{2}{3}x(x-1)^{2/3}dx+c$$

$$\Rightarrow y^{2/3}\ (x-1)^{2/3}=\int\frac{2}{3}x(x-1)^{2/3}dx+c$$

$$\Rightarrow y^{2/3}\ (x-1)^{2/3}=\frac{2}{3}\left[\frac{x(x-1)^{5/3}}{\frac{5}{3}}-\int\frac{(x-1)^{5/3}dx}{\frac{5}{3}}\right]+C$$

$$=\frac{2}{5}x(x-1)^{5/3}-\frac{2}{5}\frac{(x-1)^{8/3}}{\frac{8}{3}}+C$$

$$\Rightarrow y^{2/3}\ (x-1)^{2/3}=\frac{2}{5}x(x-1)^{5/3}-\frac{3}{20}(x-1)^{8/3}+C$$

$$\Rightarrow y^{2/3}\ =\frac{2}{5}x(x-1)-\frac{3}{20}(x-1)^{2}+C(x-1)^{-2/3}$$

OR

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$$

Let $\tan y=t\Rightarrow \sec^2 y\ \frac{dy}{dx}\ =\ \frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx}+2tx=x^3$$

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$$

This is a linear differential equation of the form, $\frac{dt}{dx}+Pt=Q$,

where, $P=2x$ and $Q=x^3$

Therefore, integrating factor :

$IF=e^{\int Pdx}=e^{\int 2xdx}=e^{x^2}$

Solution of the differential equation is given by

$t\,e^{x^2}=\int x^3e^{x^2}dx+C....(1)$

To solve $\int x^3e^{x^2}dx$

Let $x^2=z\Rightarrow 2x\ dx=dz$

$$\Rightarrow\int x^3e^{x^2}dx=\frac{1}{2}\int ze^zdz$$
$$=\frac{1}{2}\left[ze^z-\int e^zdz\right]+C$$
$$=\frac{1}{2}\left[ze^z-e^z\right]+C$$
$$=\frac{1}{2}\left(x^2-1\right)e^{x^2}+C....(2)$$

Thus, we have,

$$t\,e^{x^2}=\frac{1}{2}\left(x^2-1\right)e^{x^2}+C\ \left[\text{from equations (1) and (2)}\right]$$
$$\Rightarrow t=\frac{1}{2}\left(x^2-1\right)+Ce^{-x^2}$$
$$\Rightarrow \tan y=\frac{1}{2}\left(x^2-1\right)+Ce^{-x^2}\quad\left[\because t=\tan\ y\right]$$

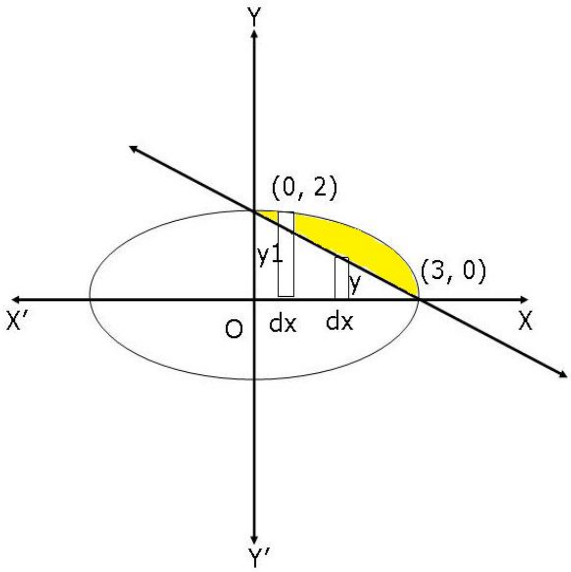
35.

Given ellipse

$$\Rightarrow\frac{x^2}{9}+\frac{y^2}{4}=1$$
$$\Rightarrow y=\frac{2}{3}\sqrt{9-x^2}$$

Given line $\frac{x}{3}+\frac{y}{2}=1$

$$\Rightarrow y=\left(2-\frac{2x}{3}\right)$$



Required Area = $\int_0^3 (y_1 - y_2)\, dx$

$$= \int_0^3 \left[\frac{2}{3}\sqrt{9-x^2} - \left(2 - \frac{2x}{3}\right)\right] dx$$

$$= \left[\frac{2}{3}\left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right) - 2x + \frac{x^2}{3}\right]_0^3$$

$$= \left[\frac{2}{3}\left(\frac{9}{2}\sin^{-1}1\right) - 6 + 3\right] - 0$$

$$= 3 \times \frac{\pi}{2} - 3 = \frac{3}{2}(\pi - 2) \text{ square units}$$