5. SURDS AND LOGARITHMS

A. Surds

1. Introduction

A surd is an irrational number. In general, if x is rational, n is positive integer and if $\sqrt[n]{x}$ is irrational, then $\sqrt[n]{x}$ is called a surd of n^{th} power. Here x is called radicand, $\sqrt[n]{x}$ is called radical sign and the index 'n' is called order of the surd. $\sqrt[n]{x}$ is read as n^{th} root of x and can be written as $x^{1/n}$ $\pm\sqrt[n]{x}$ are called simple surds.

If it is a surd of n^{th} order, then

(i) When n = 2, it is called quadratic surd.

(ii) When n = 3, it is called cubic surd.

When n = 4, it is called biquadratic surd.

Note: - Every surd is an irrational number but every irrational number is not a surd. So the representation of monomial surd on a number line is same that of irrational numbers.

e.g.,

(i) is a surd and e is irrational number (ii) π is an irrational number but it is not a surd

2. Types of Surds

(i) Pure Surd:

A surd which has unity only as rational factor the other factor being irrational is called Pure Surd. e.g. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[4]{5}$

(ii) Mixed Surd:

A surd consisting of the product of a rational and irrational is called Mixed Surd

e.g., $5\sqrt{3}$, $\sqrt{12}$,and if a is rational number and not equal to zero and $\sqrt[n]{b}$ is a surd, then a$\sqrt[n]{b}$, are mixed surd. If a = 1 they are called pure surd. Mixed Surd can be written as Pure Surd.

(iii) Compound Surd:

A surd which is the sum or difference of two or more surds is called Compound Surd.

e.g., $2 + \sqrt[3]{3}$, $1+\sqrt{2} - \sqrt{3}$

(iv) Monomial Surd: A surd consisting only one surd is called Monomial Surd.

e.g., $3\sqrt{5}$, $5\sqrt{7}$

(v) Binomial Surd: A compound surd consisting of two surds is called a Binomial Surd.

e.g. $\sqrt{2}+3\sqrt{3}, \sqrt{3}-\sqrt{7}$

(vi) Trinomial Surd: A compound surd consist of 3 surds is called Trinomial Surd.

e.g. $\sqrt{7} + \sqrt{5} - \sqrt{3}$,$3\sqrt{5} - 4\sqrt{2} - 2\sqrt{11}$

(vii) Similar Surds: If two surds are different multiples of the same surd. They are called Similar Surds otherwise they are Dissimilar Surds.

e.g., $2\sqrt{2}, 5\sqrt{2}$ are similar surds and $3\sqrt{3}$, $6\sqrt{5}$ are dissimilar surds

(viii) Conjugate Surd: Two conjugate surds which are differ only in signs (+/–) between them

e.g., $a + \sqrt{b}$ and $a - \sqrt{b}$ are Conjugate Surds. Sometimes conjugate and reciprocal are same

e.g., $2 - \sqrt{3}$ is conjugate of $2 + \sqrt{3}$ and reciprocal of $2 - \sqrt{3}$ is $2 + \sqrt{3}$

3. Rationalization of Surds

The process of converting a surd to a rational number by multiplying it with a suitable Rationalising Factor.

3.1 Rationalising Factor

When the product of two surds is a rational number, then each surd is called Rationalizing Factor (R.F.)

e.g., $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1$ which is rational

3.1.1 R.F. of Monomial Surd

One of R.F. of $a^{1/n}$ is $a^{\left(1-\frac{1}{n}\right)}$

e.g., $5^{3/5}$ and $5^{2/5}$ are Rationalising Factor of each other

3.1.2 R.F. of Binomial Surd

R.F. of $(a + \sqrt{b})$ is $(a - \sqrt{b})$ and that of $\sqrt{a} - \sqrt{b}$ is $\sqrt{a} + \sqrt{b}$.

3.1.3 R.F. of Trinomial Surd

R.F. of $\left[\left(\sqrt{a} + \sqrt{b}\right) - \sqrt{c}\right]$ is $\left[\left(\sqrt{a} + \sqrt{b}\right) + \sqrt{c}\right]$

4. Some Important Results

(i) If a + $\sqrt{b}$ = c + $\sqrt{d}$ where a, c are rational number and $\sqrt{b}$, $\sqrt{d}$ are surds, then a = c and b = d

(ii) If a, b, $\sqrt{a^2-b}$ are positive rational numbers and $\sqrt{b}$ is a surd, then

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

(iii) If a, b, c, d are positive rational numbers and $\sqrt{b},\sqrt{c},\sqrt{d}$ are surds then

$$\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}} = 4\sqrt{\frac{bd}{4c}} + 4\sqrt{\frac{bc}{4d}} + 4\sqrt{\frac{cd}{4b}}$$

(iv) $5\sqrt{a+k\sqrt{b}} = \sqrt{\sqrt{\frac{4b^2+k}{5}}} - b + \sqrt{b}$

(v) $3\sqrt{a+b\sqrt{c}} = \sqrt{\frac{b-c}{3}} + \sqrt{c}$

(vi) $\sqrt[3]{a}+\sqrt[3]{b}$ is a R.F. of $a^{2/3} - a^{1/3}b^{1/3} + b^{2/3}$ and vice versa

(vii) $\sqrt[3]{a}-\sqrt[3]{b}$ is R.F. of $a^{2/3} + a^{1/3}b^{1/3} + b^{2/3}$ and vice versa

(viii) $(a+\sqrt{b})^{x^2-k} + (a-\sqrt{b})^{x^2-k} = 2a$, $a^2 - b = 1$ $\Rightarrow x = \pm\sqrt{k\pm 1}$

5. Algebra of Surds

5.1 Law of Surds and Exponents

If a > 0, b > 0 and n is a positive rational number then

(i) $\left(\sqrt[n]{a}\right)^n = \left(a^{\frac{1}{n}}\right)^n = a$

(ii) $\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{ab}$ [Here order should be same]

(iii) $\sqrt[n]{a}\div\sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$

(iv) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[m]{\sqrt[n]{a}}$

(v) $\sqrt[n]{a} = \sqrt[n\times p]{a^p}$, $\sqrt[n]{a^p} = a^{p/n}$ or, $\sqrt[n]{a^m} = \sqrt[n\times p]{a^{m\times p}} = \sqrt[n]{\sqrt[p]{(a^m)^p}}$ [Important for changing order of surds]

(vi) $\sqrt[m]{a}\times\sqrt[n]{a} = a^{\frac{1}{m}}\times a^{\left(\frac{1}{3}+\frac{1}{n}\right)} = a^{\frac{m+n}{mn}} = \sqrt[mn]{a^{m+n}}$

(vii) $\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}} = \dfrac{a^{\frac{1}{m}}}{a^{\frac{1}{n}}} = a^{\frac{1}{m}-\frac{1}{n}} = a^{\frac{n-m}{mn}} = \sqrt[mn]{a^{n-m}}$

(viii) If $a^n = b$ then $a = b^{\frac{1}{n}} \Rightarrow a = \sqrt[n]{b}$

(ix) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

5.2 Comparison of Surds

If two surds are of same order then one whose radicand is larger is the larger of the two or if $x > y > 0$ and $n > 1$ is + ve integer then $\sqrt[n]{x} > \sqrt[n]{y}$

e.g., $\sqrt[3]{19} > \sqrt[3]{13}$, $\sqrt[7]{18} < \sqrt[7]{93}$

5.3 Identification of Surds

(i) $\sqrt[4]{25}$ is a surd as radicand is a rational number.

Similar examples $\sqrt[3]{5}$, $\sqrt[4]{12}$, $\sqrt[5]{7}$, $\sqrt{12}$, ...

(ii) $\sqrt{3}+1$ is a surd (as surd + rational number will give a surd)

Similar examples $3-\sqrt{2}$, $2+\sqrt{3}$, $\sqrt[3]{3}+1$, ...

(iii) $\sqrt{9-4\sqrt{5}}$ is a surd as $9-4\sqrt{5}$ is a perfect square of $\left(2-\sqrt{5}\right)$.

Similar examples $\sqrt{7+4\sqrt{3}}, 7-4\sqrt{3}, \sqrt{9+4\sqrt{5}}$,....

(iv) $\sqrt[3]{\sqrt[3]{(5)^{\frac{1}{2}}}}$ is a surd as $\sqrt[3]{((5)^{1/2})^{1/3}} = \left(5^{\frac{1}{6}}\right)^{\frac{1}{3}} = 5^{\frac{1}{18}} = \sqrt[18]{5}$

Similar examples $\sqrt[3]{\sqrt{3}}, \sqrt[4]{\sqrt[5]{6}}$,.....

(v) These are not a surd:

(a) $\sqrt{9} = 3$ is not a surd.

(b) $\sqrt{1+\sqrt{5}}$, because $1+\sqrt{5}$ is not a perfect square.

(c) $\sqrt[3]{3+\sqrt{2}}$, because radicand is an irrational number.

Illustration 1: If $x = 3+3^{1/3}+3^{2/3}$, then find the value of $x^3-9x^2+18x-12$.

Sol: $x = 3+3^{1/3}+3^{2/3}$

$\Rightarrow x-3 = 3^{1/3}+3^{2/3}$

Cubing both sides

$(x-3)^3 = \left(3^{1/3}+3^{2/3}\right)^3$

$\Rightarrow x^3-9x^2+27x-27 = 12+3(3)(x-3)$ since $\left[3^{1/3}+3^{2/3} = x-3\right]$

$\Rightarrow x^3-9x^2+18x-12 = 0$

Illustration 2: If $a^x = m$, $a^y = n$ and $a^2 = (m^y.n^x)^z$ then find the value of xyz.

Sol: Given $a^2 = (m^y.n^x)^z$

$\Rightarrow a^2 = [(a^x)^y.(a^y)^x]^z \qquad [\therefore m = a^x \;, n = a^y]$

$\Rightarrow a^2 = [a^{xy}.a^{xy}]^z$

$\Rightarrow a^2 = [a^{2xy}]^z$

$\Rightarrow a^2 = a^{2xyz}$

Here base is same

Hence 2 = 2xyz $\qquad \Rightarrow$ xyz = 1.

Illustration 3: Simplify $\sqrt[3]{3} \times \sqrt[4]{2}$.

Sol: LCM of 3 and 4 is 12

$3^{1/3} = 3^{4/12} = \sqrt[12]{3^4}$ And $2^{1/4} = 2^{3/12} = \sqrt[12]{2^3}$

$\sqrt[3]{3} \times \sqrt[4]{2} = \left(\sqrt[12]{3^4}\right)\left(\sqrt[12]{2^3}\right) = \sqrt[12]{(3^4)(2^3)} = \sqrt[12]{81 \times 8} = \sqrt[12]{648}$

Illustration 4: Arrange $\sqrt[4]{6}, \sqrt[3]{7}$ and $\sqrt{5}$ in ascending order.

Sol: L.C.M. of 4, 3, 2 is 12.

$\therefore \sqrt[4]{6} = 6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}$

$\sqrt[3]{7} = 7^{\frac{1}{3}} = 7^{\frac{4}{12}} = 12\sqrt{7^4} = \sqrt[12]{2401}$

$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{6}{12}} = \sqrt[12]{5^6} = \sqrt[12]{15625}$

Hence ascending order i.e. $\sqrt[4]{6}, \sqrt[3]{7}, \sqrt{5}$

B. Logarithms

1. Definition

If 'a' is a positive real number, not equal to 1 and x is a rational number such that $a^x = N$, then x is the Logarithm of N to the base a.

$\therefore$ If $a^x = N$ then $\log_a N = x$. [Remember N will be + ve i.e., $N \neq 0$]

e.g., $2^3 = 8$ then $\log_2 8 = 3$

2. System of Logarithms

There are two systems which are general used Napierian Logarithms and Common Logarithms

2.1 Napierian Logarithms

The logarithms of numbers calculated to the base 'e' are called Natural Logarithms or Napierian Logarithms. Here "e" is an irrational number lying between 2 and 3 (Approx value of e = 2.73)

2.2 Common Logarithms

Logarithms to the base 10 are called Common Logarithms.

PLANCESS CONCEPTS

$\log_{10} 1 = 0$

$\log_{10} 10 = 1$

$\log_{10} 100 = 2$

$\log_{10} 1000 = 3$

$\log_{10} 10000 = 4$

$\log_{10} 100000 = 5$

$\log_{10} 1000000 = 6$

So you can see, the base-10 log of a number tells you approximately its order of magnitude. This is how logarithms can be very useful. It helps us to convert a large number to a very small one and at the same time the smaller numbers can be converted to a number which can be comparable to the bigger ones.

USE OF LOGARITHMS IN OUR LIFE

An **earthquake** is what happens when two blocks of the earth suddenly slip past one another. The amount of energy released during an Earthquake can be enormous. Richter Scale is used to study the intensity of the earthquakes. Because of the huge range of the energy released from the earthquakes, the knowledge of logarithms turns out to be very helpful. In elementary terms, the Richter Scale is nothing but a base -10 logarithmic scale. This implies that it describes the energy released in terms of the order of the magnitude instead of its original value.

- **Magnitude wise impact of the earthquakes:**
- **Magnitude 3 and lower –** are almost imperceptible or weak causing no damage
- **Magnitude 5 –** it can be felt by everyone and can cause slight damage to normal buildings.
- **Magnitude 7 –** can cause serious damage over larger areas, (depending on the depth of the epicenter).
- **Magnitude 9 and above –** Total destruction, severe damage, death toll usually over 50,000.

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3. Properties of Logarithms

(i) Logarithms are defined only for positive real numbers

(ii) Logarithms are defined only for positive bases different from 1.

(iii) In $\log_b a$, neither a nor b is negative i.e., log of (–) ve number not defined but the value of $\log_b a$ can be negative **e.g.,** $10^{-2} = 0.01$, $\log_{10} 0.01 = -2$

(iv) log of 0 is not defined as $a^n = 0$ not possible

(v) log of 1 to any base is 0. **e.g.,** $\log_2 1 = 0$ ($\therefore\ 2^0 = 1$)

log of a number to the same base is 1. **e.g.,** $\log_4 4 = 1$

Logarithms of the same number to different base have different values. i.e., if $m \neq n$ then

$\log_m a \neq \log_n a$. In other words, if $\log_m a = \log n a$ then $m = n$.

e.g., $\log_2 16 = \log_n 16 \Rightarrow n = 2$, $\log_2 16 \neq \log_4 16$. Here $m \neq n$

Logarithms of different numbers to the same base are different i.e., if $a \neq b$, then $\log_m a \neq \log_m b$.

In other words if $\log_m a = \log_m b$ then $a = b$

e.g., $\log_{10} 2 \neq \log_{10} 3 \quad \therefore\ a \neq b$

$\log_{10} 2 = \log_{10} y \Rightarrow y = 2 \ \therefore\ a = b$

4. Fundamental Laws of Logarithms

Logarithm to any base a (where a > 0 and a ≠ 1).

(i) $\log_a (mn) = \log_a m + \log_a n$ [Where m and n are +ve numbers]

(ii) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log a^m = m \log a$

(iv) $\log a^m = \frac{\log_b m}{\log_b a}$ (change of base rule)

(v) $\log_b a \times \log_c b = \log_c a$

(vi) $\log_a m . \log_m a = 1$

(vii) If ‘a’ is a positive real number and ‘n’ is a positive rational number, then $a^{\log_a n} = n$

(viii) If ‘a’ is a positive real number and ‘n’ is a positive rational number, then $\log_{a^q} n^p = \frac{p}{q} \log_a n$

(ix) $\log_{a^n} m = \frac{1}{n} \log_a m$

(x) $p^{\log_a q} = p^{\log_a p}$

(xi) $\log_a m = \frac{1}{\log_m a}$

(xii) $\log_a \left(\frac{mn}{pq}\right) = \log_a m + \log_a n - \log_a p - \log_a q$

(xiii) $e^{\log_e x} = x$

(xiv) $\log_b a \Leftrightarrow \log_c a . \log_b c$

(xv) $\log_b a = \frac{1}{2} \log_{\sqrt{b}} a$

(xvi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$

(xvii) If a ≠ 1, a > 0 then $\log_a a = 1$; $\log_e e = 1$

(xviii) $\log_a 1 = 0$ (a > 0, a ≠ 1)

(xix) ${}_a\log m = {}_m\log a$

5. Graphs of Logarithmic Functions

CASE – I : If a > 1 then $\log_a x$ is an increasing function.

(a) $\log_a x < 0$ for all x satisfying 0 < x < 1 (curve lies below x axis)

(b) $\log_a x = 0$ for x = 1

(c) $\log_a x > 0$ for x > 1 (curve lies above x axis)

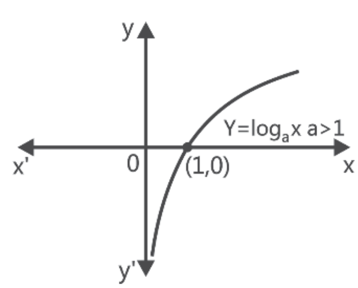

Figure 5.1

CASE – II: If $0 < a < 1$, then $\log_a x$ is a decreasing function.

(a) $\log_a x < 0$ for all $x > 1$ (curve lies below x axis)

(b) $\log_a x = 0$ for $x = 1$

(c) $\log_a x > 0$ for all x satisfying $0 < x < 1$ (curve lies above x axis)

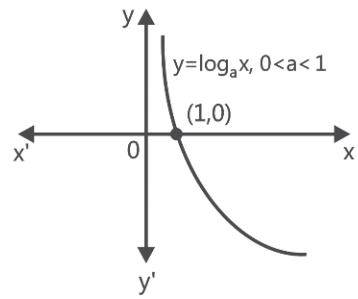

Figure 5.2

PLANCESS CONCEPTS

NOTE : 1. $x > y \Rightarrow \log_a x > \log_a y$ if $a > 1$

2. $x > y \Rightarrow \log_a x < \log_a y$ if $0 < a < 1$.

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Illustration 1: If $\log (x^2y^3) = a$ and $\log \frac{x}{y} = b$, Find log x and log y.

Sol: $\because \log (x^2y^3) = a$ and $\log \left(\frac{x}{y}\right) = b$

$\because 2 \log x + 3 \log y = a$ .....(1)

And $\log x – \log y = b$ .....(2)

Multiply (2) by 2 and subtracting from (1) we have $5 \log y = a – 2b$; $\log y = \frac{a-2b}{5}$

Again multiplying (2) by 3 and adding $5 \log x = a + 3b$; $\log x = \frac{a+3b}{5}$

Illustration 2: Find the value of $7 \log \left(\frac{16}{15}\right) + 5 \log \left(\frac{25}{24}\right) + 3 \log \left(\frac{81}{80}\right)$.

Sol: $7 \log \left(\frac{16}{15}\right) + 5 \log \left(\frac{25}{24}\right) + 3 \log \left(\frac{81}{80}\right) = 7 \log \left(\frac{2^4}{3\times5}\right) + 5\log\left(\frac{5^2}{2^3\times3}\right) + 3\log\left(\frac{3^4}{5\times2^4}\right)$

$= 7(4\log2 – \log3 – \log5) + 5 (2\log5 – 3\log2 – \log3) + 3(4\log3 – \log5 – 4\log2)$

$= 28 \log2 – 7\log3 – 7\log5 + 10 \log5 – 15\log2 – 5\log 3 + 12\log 3 – 3\log5 – 12\log2$

$= 28\log2– 27\log2 = \log2$

Illustration 3: If $\log_x 256 = \frac{8}{5}$ then find x.

Sol: $\log_x 256 = \frac{8}{5} \Rightarrow x^{8/5} = 256 \Rightarrow (x^{1/5})^8 = 2^8 \Rightarrow x^{1/5} = 2$ or $x = 2^5 \quad \therefore x = 32$

Illustration 4: Evaluate: $4^{2-\log_4 5}$.

Sol: Given $4^{2-\log_4 5} = 4^2.4^{-\log_4 5}$; $[= a^m.a^n] = 16.4^{\log_4 5^{-1}} = 16 \times 5^{-1} = \frac{16}{5}$

Illustration 5: Find x if $\log_x 3 + \log_x 9 + \log_x 729 = 9$.

Sol: $\log_x 3 + \log_x 9 + \log_x 729 = 9$; $\log_x 3 + \log_x 3^2 + \log_x 3^6 = 9$

$\log_x 3 + 2\log_x 3 + 6\log_x 3 = 9$; $9\log_x 3 = 9$; $\log_x 3 = \frac{9}{9} = 1$

$x^1 = 3 \quad \therefore x = 3$

6. Characteristic and Mantissa

The integral part of a logarithm is called characteristic and decimal part is called Mantissa.

To find characteristic:

Case1: If the number is greater than unity and if there are n digits in integral part, then its characteristic = (n – 1)

Case2: If the number is less than unity and if there are n zeroes after decimal (and number starts), then its characteristic is $\overline{n+1}$ [called as Bar (n + 1)].

Note: $\bar{6}.325$ means – 6 + 0.325 whereas – (6.325) means – 6.325.

(i) $\log_b a = \frac{\log a}{\log b}$ (ii) $\log_c a.\log_b c.\log_d b = \log_d a$

e.g. $\log_2 x.\log_3 2.\log\log_4 3 ... \log_{n+1} n = \log_{n+1} x$

(i) $\frac{1}{\log_{x^n}(xyz)} + \frac{1}{\log_{y^n}(xyz)} + \frac{1}{\log_{z^n}(xyz)} = n$

e.g. $\frac{1}{\log_{x^2}(xyz)} + \frac{1}{\log_{y^2}(xyz)} + \frac{1}{\log_{z^2}(xyz)} = 2$

SUMMARY

- A surd is an irrational number. In general, if x is rational, n is positive integer and if $\sqrt[n]{x}$ is irrational, then $\sqrt[n]{x}$ is called a surd of n^{th} power.
- **Rationalizing Factor:** When the product of two surds is a rational number, then each surd is called Rationalizing Factor (R.F.)
- **Law of Surds and Exponents**

  If a > 0, b > 0 and n is a positive rational number then,

  (i) $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n} = a$ (ii) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ [Here, order should be same]

  (i) $\sqrt[n]{a} \div \sqrt[n]{b} = \sqrt[n]{\frac{a}{b}}$

  (ii) $\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a} = \sqrt[m]{\sqrt[n]{a}}$

  (iii) $\sqrt[n]{a} = \sqrt[n\times p]{a^p}$, $\sqrt[n]{a^p} = a^{p/n}$ or $\sqrt[n]{a^m} = \sqrt[n\times p]{a^{m\times p}} = \sqrt[n]{\sqrt[p]{(a^m)^p}}$

  [Important for changing order of surds]

  (i) $\sqrt[m]{a} \times \sqrt[n]{a} = \sqrt[mn]{a^{m+n}}$

  (ii) $\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}} = \sqrt[mn]{a^{m-n}}$

  (iii) If $a^n = b$ then $a = b^{\frac{1}{n}} \Rightarrow a = \sqrt[n]{b}$

  (iv) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

- **Comparison of Surds:** If two surds are of same order then one whose radicand is larger is the larger of the two or if x > y > 0 and n > 1 is + ve integer then $\sqrt[n]{x} > \sqrt[n]{y}$
- If 'a' is a positive real number, not equal to 1 and x is a rational number such that $a^x = N$, then x is the Logarithm of N to the base a.

  ∴ If $a^x = N$ then $\log_a N = x$. [Remember N will be + ve i.e., $N \neq 0$]
- **Napierian Logarithms:** The logarithms of numbers calculated to the base 'e' are called Natural Logarithms or Napierian Logarithms.
- **Common Logarithms:** Logarithms to the base 10 are called Common Logarithms.
- **Properties of Logarithms**

  (i) Logs are defined only for positive real numbers.

  (ii) Logs are defined only for positive bases different from 1.

(iii) In $\log_b a$, neither a nor b is negative i.e., log of (–) ve number not defined but the value of $\log_b a$ can be negative

Fundamental Laws of Logarithms

Logarithm to any base a (where a > 0 and a ≠ 1).

(i) $\log_a(mn) = \log_a m + \log_a n$ [Where m and n are +ve numbers]

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m = m \log a$

(iv) $\log_a m = \frac{\log_b m}{\log_b a}$ (change of base rule)

(v) $\log_b a \log_c b = \log_c a$

(vi) $\log_a m . \log_m a = 1$

(vii) If 'a' is a positive real number and 'n' is a positive rational number, then $a^{\log_a n} = n$

(viii) If 'a' is a positive real number and 'n' is a positive rational number, then $\log_{a^q} n^p = \frac{p}{q}\log_a n$

(ix) $\log_{a^n} m = \frac{1}{n}\log_a m$

(x) $p^{\log_a q} = p^{\log_a p}$

(xi) $\log_a m = \frac{1}{\log_m a}$

(xii) $\log_a\left(\frac{mn}{pq}\right) = \log_a m + \log_a n - \log_a p - \log_a q$

(xiii) $e^{\log_e x} = x$

(xiv) $\log_b a \Leftrightarrow \log_c a . \log_b c$

(xv) $\log_b a = \frac{1}{2}\log_{\sqrt{b}} a$

(xvi) $\log_{a^n} x^m = \frac{m}{n} \log_a x$

(xvii) If a ≠ 1, a > 0 then $\log_a a = 1$; $\log_e e = 1$

(xviii) $\log_a 1 = 0$ (a > 0, a ≠ 1)

(xix) $_a\log m = {}_m\log a$

Graphs of Logarithmic Functions

CASE – I: If a > 1

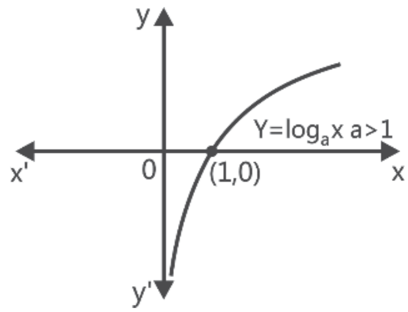

Figure 5.3

CASE – II: If 0 < a < 1, then

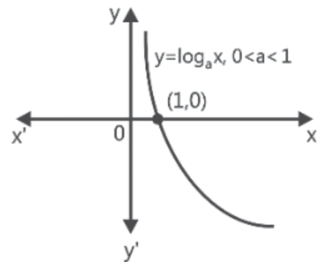

Figure 5.4

Characteristic and Mantissa

The integral part of a logarithm is called Characteristic and decimal part is called Mantissa.

SOLVED EXAMPLES

Example 1: Find the value of $\frac{\log_{10} 6+1}{\log_{10} 60}$.

Sol: $\frac{\log_{10} 6+1}{\log_{10} 60}=\frac{\log_{10} 6+\log_{10} 10}{\log_{10} 60}=\frac{\log_{10}(6\times 10)}{\log_{10} 60}=\frac{\log_{10} 60}{\log_{10} 60}=1.$

Example 2: Prove that $(\log x)^2 - (\log y)^2 = (\log xy)\left(\log \frac{x}{y}\right)$

Sol: $(\log x)^2 - (\log y)^2$ is in the form $(a^2 - b^2)$

We know that $a^2 - b^2 = (a + b)(a - b)$

$\therefore (\log x)^2 - (\log y)^2 = (\log x + \log y)(\log x - \log y) = (\log xy)\left(\log\frac{x}{y}\right)$

$\therefore (\log x)^2 - (\log y)^2 = (\log xy)\left(\log\frac{x}{y}\right)$

Example 3: If $(3.5)^x = (0.035)^y = 10^3$, find $\frac{1}{x}-\frac{1}{y}$.

Sol: $(3.5)^x = 10^3 \Rightarrow \log_{3.5} 1000 = x$

$(0.035)^y = 10^3 \Rightarrow \log_{0.035} 1000 = y$

$\frac{1}{x}-\frac{1}{y}=\log_{1000} 3.5-\log_{1000} 0.035=\log_{1000}\frac{3.5}{0.035}=\log_{1000} 100=\log_{10^3}(10)^2=\frac{2}{3}$

Example 4: Solve: $\log(3 + 2\log(1 + x)) = 0$.

Sol: $\log(3 + 2\log(1 + x)) = \log 1$

$3 + 2\log(1 + x) = 1$

$2\log(1 + x) = -2$

$\log(1 + x) = -1$

$\log_{10}(1 + x) = -1$

$\Rightarrow 1 + x = 10^{-1}$

$\Rightarrow x = \frac{1}{10}-1=\frac{-9}{10}$

Example 5: Solve: $5^{\log x} + x^{\log 5} = 50$.

Sol: $5^{\log x} = x^{\log 5}$

$\Rightarrow 5^{\log x} + x^{\log 5} = 50$

$\Rightarrow 5^{\log x} + 5^{\log x} = 50$

$\Rightarrow 2(5^{\log x}) = 50$

$\Rightarrow 5^{\log x} = 25$

$\Rightarrow 5^{\log x} = 5^2$

$\Rightarrow \log x = 2$

$x = 10^2; x = 100.$

Example 6: If a, b, c are three consecutive positive integers, then show that $2\log b = \log(1 + ac)$.

Sol: Let the three consecutive positive integers a, b, c be of the form $m - 1$, m, $m + 1$ respectively, then $a = m - 1$, $b = m$, $c = m + 1$

Now, taking $\log(1 + ac) = \log[1 + (m - 1)(m + 1)] = \log(1 + m^2 - 1) = \log m^2 = 2\log m$

But $b = m$

$\Rightarrow \log(1 + ac) = 2\log m = 2\log b$

$\therefore 2\log b = \log(1 + ac)$

Example 7: Find the value of

$$\log_{1003}\left(1+\frac{1}{2}\right)+\log_{1003}\left(1+\frac{1}{3}\right)+....+\log_{1003}\left(1+\frac{1}{2005}\right)$$

Sol: $\log_{1003}\frac{3}{2}+\log_{1003}\frac{4}{3}+....+\log_{1003}\frac{2006}{2005}$

$$\log_{1003}\left(\frac{3}{2}\times\frac{4}{3}....\frac{2005}{2004}\times\frac{2006}{2005}\right)$$

$$\log_{1003}\frac{2006}{2} = \log_{1003}1003 = 1$$

Example 8: Solve: $\log_4\log_2\log_{\sqrt{2}}\log_3(x-2006) = 0.$

Sol: $\log_2\log_{\sqrt{2}}\log_3(x-2006) = 4^0 = 1$

$\log_{\sqrt{2}}\log_3(x-2006) = 2$

$\log_3(x-2006) = (\sqrt{2})^2 = 2$

$x - 2006 = 3^2$

$x = 2006 + 9 = 2015$

Example 9: If $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b'}$ then find $a^a b^b c^c$.

Sol: Let $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}=k$

$\log a = k(b - c) = kb - kc$

$\log b = k(a - c) = ka - kc$

$\log c = k(a - b) = ka - kb$

$\log a^a = a \log a = kab - kac$ … (1)

$\log b^b = b \log b = kbc - kba$ … (2)

$\log c^c = c \log c = kca - kcb$ … (3)

(1) + (2) + (3) = 0

$\log_{10} a^a\, b^b\, c^c = 0$

$a^a b^b c^c = 10^\circ = 1.$

Example 10: Evaluate: $(-32)^{\frac{2}{5}}$.

Sol: $(-32)^{\frac{2}{5}} = \left[(-2)\times(-2)\times(-2)\times(-2)\times(-2)\right]^{\frac{2}{5}}$

$\left[(-2)^5\right]^{\frac{2}{5}} = (-2)^{5\text{x}\frac{2}{5}}$. $\left[(a^m)^n = a^{mn}\right]$

$= (-2)^2 = (-2)\times(-2) = 4$

Example 11: Evaluate: (i) $5^{-4}\text{x}(125)^{\frac{5}{3}} \div (25)^{-\frac{1}{2}}$ (ii) $\left(\frac{27}{125}\right)^{\frac{2}{3}} \times \left(\frac{9}{25}\right)^{-\frac{3}{2}}$.

Sol: (i) $5^{-4}\times(125)^{\frac{5}{3}} \div (25)^{-\frac{1}{2}} = 5^{-4}\times(5\times5\times5)^{\frac{5}{3}} \div (5\times5)^{-\frac{1}{2}}$

$= 5^{-4}\times\left(5^3\right)^{\frac{5}{3}} \div (5^2)^{-\frac{1}{2}} = 5^{-4}\times(5)^{3\times\frac{5}{3}} \div (5)^{2\times\left(-\frac{1}{2}\right)}$

$= 5^{-4}\times(5)^5 \div (5)^{-1} = \frac{5^{-4}\times5^5}{5^{-1}} = 5^{-4+5-(-1)} = 5^{-4+5+1} = 5^2 = 5\times5 = 25$

(ii) $\left(\frac{27}{125}\right)^{\frac{2}{3}} \times \left(\frac{9}{25}\right)^{-\frac{3}{2}} = \left(\frac{3\times3\times3}{5\times5\times5}\right)^{\frac{2}{3}} \times \left(\frac{3\times3}{5\times5}\right)^{-\frac{3}{2}}$

$= \left[\left(\frac{3}{5}\right)^3\right]^{\frac{2}{3}} \times \left[\left(\frac{3}{5}\right)^2\right]^{-\frac{3}{2}} = \left(\frac{3}{5}\right)^{3\times\frac{2}{3}} \times \left(\frac{3}{5}\right)^{2\times\left(\frac{-3}{2}\right)}$

$= \left(\frac{3}{5}\right)^2 \times \left(\frac{3}{5}\right)^{-3} = \left(\frac{3}{5}\right)^{2+(-3)}$ $\left[a^m \times a^n = a^{m+n}\right]$

$= \left(\frac{3}{5}\right)^{2-3} = \left(\frac{3}{5}\right)^{-1} = \frac{1}{\frac{3}{5}} = 1\times\frac{5}{3} = \frac{5}{3}$

Example 12: If $1960 = 2^a.5^b.7^c$, calculate the value of $2^{-a}.7^b.5^{-c}$.

Sol: $1960 = 2^a.5^b.7^c \Rightarrow 2^a.5^b.7^c = 1960$

$\Rightarrow 2^a.5^b.7^c = 2\times2\times2\times5\times7\times7$

$\Rightarrow 2^a.5^b.7^c = 2^3\times5^1\times7^2$ ....(i)

Comparing powers of 2,5 and 7 on both sides of equation (i), we get

a=3

b=1

c=2

Hence value of:

$$2^{-a}.7^b.5^{-c} = 2^{-3}.7^1.5^{-2} = \frac{1}{2^3}\times7\times\frac{1}{5^2} = \frac{7}{2^3\times5^2} = \frac{7}{2\times2\times2\times5\times5} = \frac{7}{8x25} = \frac{7}{200}$$

Example 13: Show that:

$$\left(\frac{a^m}{a^{-n}}\right)^{m-n}\times\left(\frac{a^n}{a^{-l}}\right)^{n-l}\times\left(\frac{a^l}{a^{-m}}\right)^{l-m} = 1$$

Sol: L.H.S= $\left(\frac{a^m}{a^{-n}}\right)^{m-n}\times\left(\frac{a^n}{a^{-l}}\right)^{n-l}\times\left(\frac{a^l}{a^{-m}}\right)^{l-m}$

$\Rightarrow \left[a^{m-(-n)}\right]^{m-n}\times\left[a^{n-(-l)}\right]^{n-l}\times\left[a^{l-(-m)}\right]^{l-m}$

$\Rightarrow \left[a^{m+n}\right]^{m-n}\times\left[a^{n+l}\right]^{n-l}\times\left[a^{l+m}\right]^{l-m}$

$\Rightarrow a^{(m+n)(m-n)} x\ a^{(n+l)(n-l)} x\ a^{(l+m)(l-m)}$

$\Rightarrow a^{m^2-n^2}\times a^{n^2-l^2}\times a^{l^2-m^2}. \Rightarrow \left[A^2-B^2=(A+B)(A-B)\right]$

$\Rightarrow a^{m^2-n^2+n^2-l^2+l^2-m^2}$

$\Rightarrow a^0$

$\Rightarrow$ L.H.S = 1 = R.H.S

EXERCISE 1 – For School Examinations

True / False

Directions: Read the following statements and write your answer as true or false.

Q.1. (i) If $\log_{10} x = a$, then $10^x = a$ (ii) If $x^y = z$, then $y = \log_z x$ (iii) $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$

Short Answer Questions

Directions: Give answer in two to three sentences.

Q.2. (i) $\log 1 \text{ x } \log 1000 = 0$ (ii) $\frac{\log x}{\log y} = \log x - \log y$

(iii) If $\frac{\log 25}{\log 5}$ =log x, then x = 2 (iv) $\log x \times \log y = \log x - \log y$

Q.3. Simplify:

(i) $\left(8x^3 \div 125y^3\right)^{\frac{2}{3}}$ (ii) $(a+b)^{-1}.(a^{-1}+b^{-1})$

(iii) $\frac{5^{n+3} - 6 \times 5^{n+1}}{9 \times 5^n - 5^n x 2^2}$ (iv) $\left(3x^2\right)^{-3} \times \left(x^9\right)^{\frac{2}{3}}$

Q.4. Evaluate: (i) $\sqrt{\frac{1}{4}} + (0.01)^{-\frac{1}{2}} - (27)^{\frac{2}{3}}$ (ii) $\left(\frac{27}{8}\right)^{\frac{2}{3}} - \left(\frac{1}{4}\right)^{-2} + 5^0$

Q.5. Simplify each of the following and express with positive index:

(i) $\left[1 - \left\{1 - \left(1-n\right)^{-1}\right\}^{-1}\right]^{-1}$

Long Answer Questions

Directions: Give answer in four to five sentences.

Q.6. If $a = x^{m+n}.y^{l}$; $b = x^{n+l}.y^n$ and $c = x^{l+m}.y^n$, prove that: $a^{m+n}.b^{n+l}.c^{l-m} = 1$

Q.7. Solve: (i) $8 \times 2^{2x} + 4 \times 2^{x+1} = 1 + 2^x$ (ii) $\left(\sqrt{3}\right)^{x-3} = \left(\sqrt[4]{3}\right)^{x+1}$

Q.8. Prove that:

(i) $\left(\frac{x^a}{x^b}\right)^{a+b-c} \left(\frac{x^b}{x^c}\right)^{b+c-a} \left(\frac{x^c}{x^a}\right)^{c+a-b} = 1$ (ii) $\left(\frac{x^{a(b-c)}}{x^{b(a-c)}}\right) \div \left(\frac{x^b}{x^a}\right)^c = 1$

Q.9. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that: $\frac{2xz}{x+z}$

Q.10. If $5^{-p} = 4^{-q} = 20^r$; show that: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 0$.

Q.11. Solve $\frac{3^{n+1}}{3^{n(n-1)}} \div \frac{9^{n+1}}{\left(3^{n+1}\right)^{(n-1)}}$

Q.12. Express in terms of log 2 and log 3: $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243}$

Q.13. Express each of the following in a form free from logarithm:

(i) $\log V = 2\log 2 - \log 3 + \log\pi + 3\log r$

(ii) $\log F = \log G + \log m_1 + \log m_2 - 2\log d$

Q.14. Evaluate each of the following without using tables:

(i) $\log_{10} 8 + \log_{10} 25 + 2\log_{10} 4 - \log_{10} 32$ (ii) $\log 4 + \frac{1}{3}\log 125 - \frac{1}{5}\log 32$

Q.15. Prove that: $2\log\frac{15}{18} - \log\frac{25}{162} + \log\frac{4}{9} = \log 2.$

Q.16. Find x, if: $x - \log 48 + 3\log 2 = \frac{1}{3}\log 125 - \log 3$

Q.17. Solve for x: $\log(x+4) - \log(x-4) = 1$

Q.18. If $\log_{10} 2 = x$ and $\log_{10} 3 = y$; express each of the following in terms of 'x' and 'y':

(i) log 12 (ii)log 2.25 (iii) $\log 3\frac{1}{8}$

Q.19. If log (a + b) = log a + log b, find a in terms of b.

Q.20. If $l = \log\frac{a^2}{bc}, m = \log\frac{b^2}{ca}$ and $n = \log\frac{c^2}{ab}$, find the value of l + m + n

Q.21. If $\log\frac{x-y}{2} = \frac{1}{2}(\log x + \log y)$, show that $x^2 + y^2 = 6xy$

Q.22. If $a^2 + b^2 = 23ab$, show that: $\log\frac{a+b}{5} = \frac{1}{2}(\log a + \log b)$

Q.23. Solve for a and b if a > 0 and b > 0: $\log ab = \log\frac{a}{b} + 2\log 2 = 2$

Q.24. If $\log_2(a+b) = \log_3(a-b) = \frac{\log 25}{\log 0.2}$, find the values of a and b.

SOLUTIONS

EXERCISE 1 – For School Examinations

True / False

1. (i) $10^a = x$ Hence, False

(ii) $\log_z x = y \Rightarrow z^y = x$ Hence, False

(iii) $2^3 = 8$

$8^{\frac{1}{3}} = 2 \Rightarrow (2x2x2)^{\frac{1}{3}} = 2$

$2^{3x\frac{1}{3}} = 2 \Rightarrow 2 = 2$ Hence, True

Short Answer Questions

2. (i) L.H.S $\log 1 \times \log 1000$

$= 0 \times 3$ $[\log 1 = 0 \text{ and } \log 1000 = 3]$

$= 0$

$\therefore$ Given statement that $\log\ 1\ x \log 1000 = 0$ is true.

(ii) R.H.S

$\log x - \log y$

$= \log \frac{x}{y}$ $\left[\log_a m - \log_a n = \log_a \frac{m}{n}\right]$

$\therefore$ Given statement that,

$\frac{\log x}{\log y} = \log x - \log y$ is false.

(iii) $\frac{\log 25}{\log 5} = \log x$

$\Rightarrow \frac{\log 5 \times 5}{\log 5} = \log x \Rightarrow \frac{\log 5^2}{\log 5} = \log x$

$\Rightarrow \frac{2\log 5}{\log 5} = \log x$ $\left[\log_a m^n = n \log_a m\right]$

$\Rightarrow 2 = \log x \Rightarrow \log x = 2$

$\therefore$ Given statement is false.

(iv) R.H.S = log x – log y

= log xy

But L.H.S = $\log x \times \log y$

$\therefore$ Given statement that,

$\log x \times \log y = \log x - \log y$ is false.

3.(i) $\left(8x^3 \div 125y^3\right)^{\frac{2}{3}} = \left(\frac{8x^3}{125y^3}\right)^{\frac{2}{3}} = \left(\frac{2x}{5y} \times \frac{2x}{5y} \times \frac{2x}{5y}\right)^{\frac{2}{3}} = \left[\left(\frac{2x}{5y}\right)^3\right]^{\frac{2}{3}} = \left(\frac{2x}{5y}\right)^2 = \frac{2x}{5y} \times \frac{2x}{5y} = \frac{4x^2}{25y^2}$

(ii) $(a+b)^{-1}.(a^{-1}+b^{-1}) = \frac{1}{(a+b)}.\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{1}{(a+b)}.\left(\frac{1\times b+1\times a}{ab}\right) = \frac{1}{(a+b)}.\frac{(b+a)}{ab} = \frac{1}{(a+b)}.\frac{(a+b)}{ab} = \frac{1}{ab}$

(iii) $\frac{5^{n+3}-6\times 5^{n+1}}{9\times 5^n - 5^n \times 2^2} = \frac{5^{n+1}.5^2 - 6\times 5^{n+1}}{9\times 5^n - 5^n \times 4} = \frac{5^{n+1}(5^2-6)}{5^n(9-4)} = \frac{5^n.5^1(25-6)}{5^n(5)} = \frac{5^n.5(19)}{5^n(5)} = 19$

(iv) $\left(3x^2\right)^{-3} \times \left(x^9\right)^{\frac{2}{3}} = \frac{1}{\left(3x^2\right)^3} \times x^{9\times\frac{2}{3}} = \frac{1}{(3)^3(x^2)^3} \times x^6 = \frac{x^6}{27\times x^6} = \frac{1}{27}$

4. (i) $\sqrt{\frac{1}{4}} + (0.01)^{-\frac{1}{2}} - (27)^{\frac{2}{3}} = \sqrt{\left(\frac{1}{2}\right)^2} + \left((0.1)^2\right)^{-\frac{1}{2}} - \left(3^3\right)^{\frac{2}{3}} = \frac{1}{2} + (0.1)^{-1} - (3)^2 = \frac{1}{2} + \frac{1}{0.1} - 3\times 3$

$= \frac{1}{2} + \frac{10}{1} - 9 = \frac{1x1 + 10\times 2 - 9\times 2}{2} = \frac{1+20-18}{2} = \frac{21-8}{2} = \frac{3}{2} = 1\frac{1}{2}$

(ii) $\left(\frac{27}{8}\right)^{\frac{2}{3}} - \left(\frac{1}{4}\right)^{-2} + 5^0 = \left(\frac{3\times3\times3}{2\times2\times2}\right)^{\frac{2}{3}} - \left(\frac{1\times1}{2\times2}\right)^{-2} + 1 \qquad \left[a^0 = 1\right]$

$= \left[\left(\frac{3}{2}\right)^3\right]^{\frac{2}{3}} - \left[\left(\frac{1}{2}\right)^2\right]^{-2} + 1 = \left(\frac{3}{2}\right)^{3x\frac{2}{3}} - \left(\frac{1}{2}\right)^{2x(-2)} + 1 = \left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^{-4} + 1 = \frac{3}{2}\times\frac{3}{2} - \frac{1}{\left(\frac{1}{2}\right)^{-4}} + 1$

$= \frac{9}{4} - \frac{1}{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}} + 1 = \frac{9}{4} - \frac{1}{\frac{1}{16}} + 1 = \frac{9}{4} - 1\times\frac{16}{1} + 1 = \frac{9}{4} - 16 + 1 = \frac{9}{4} - \frac{16}{1} + \frac{1}{1}$

$= \frac{9 - 16\times 4 + 1\times 4}{4} = \frac{9-64+4}{4} = -\frac{51}{4}$

5. $\left[1-\left\{1-(1-n)^{-1}\right\}^{-1}\right]^{-1} = \left[1-\left\{1-\frac{1}{1-n}\right\}^{-1}\right]^{-1}$

$\Rightarrow \left[1-\left\{\frac{1-n-1}{1-n}\right\}^{-1}\right]^{-1} = \left[1-\left\{\frac{-n}{1-n}\right\}^{-1}\right]^{-1} \Rightarrow \left[1-\left\{\frac{1-n}{-n}\right\}\right]^{-1} = \left[1+\frac{1-n}{n}\right]^{-1} \Rightarrow \left[\frac{n+1-n}{n}\right]^{-1} = \left[\frac{1}{n}\right]^{-1} = n$

Long Answer Questions

6. $a = x^{m+n}.y^{l}$ ....(1)

$b = x^{n+l}.y^{m}$ ....(2)

$c = x^{l+m}.y^{n}$ ....(3)

$L.H.S = a^{m+n}.b^{n+l}.c^{l-m}$ ....(4)

Putting the value of a,b,c from (n-l)(1), (2), (3) respectively in (4), we get

$L.H.S = (x^{m+n}.y^{l})^{(m-n)}.(x^{n+l}.y^{m})^{(n-l)}.(x^{l+m}.y^{n})^{(l-m)}$

$\Rightarrow (x^{m+n})^{(m-n)}.(y^{l})^{(m-n)}.(x^{n+l})^{(n-l)}.(y^{m})^{(n-l)}.(x^{l+m})^{(l-m)}.(y^{n})^{(l-m)}$

$\Rightarrow (x)^{(m+n)(m-n)}.(y)^{(l)(m-n)}.(x)^{(n+l)(n-l)}.(y)^{(m)(n-l)}.(x)^{(l+m)(l-m)}.(y)^{(n)(l-m)}$

$\Rightarrow x^{m^2-n^2}.x^{n^2-l^2}.x^{l^2-m^2}.y^{lm-nl}.y^{m-n-lm}.y^{nl-mn}$

$\Rightarrow x^{m^2-n^2+n^2-l^2+l^2-m^2}.y^{lm-nl+m-n-lm+nl-mn}$

$\Rightarrow L.H.S = x^{m^2-n^2+n^2-l^2+l^2-m^2}.y^{lm-nl+m-n-lm+nl-mn}$

$\Rightarrow x^{0}.y^{0}$ $\Rightarrow 1.1 = 1 = R.H.S$

7. (i) $8 \times 2^{2x} + 4 \times 2^{x+1} = 1 + 2^{x}$

$\Rightarrow 8 \times \left(2^{x}\right)^{2} + 4 \times 2^{x} \text{ x } 2^{1} = 1 + 2^{x}$

$\Rightarrow 8 \times \left(2^{x}\right)^{2} + 8 \times 2^{x} = 1 + 2^{x}$

$\Rightarrow 8 \times \left(2^{x}\right)^{2} + 8 \times 2^{x} - 2^{x} - 1 = 0$

$\Rightarrow 8 \times \left(2^{x}\right)^{2} + 2^{x}(8-1) - 1 = 0$

$\Rightarrow 8\left(2^{x}\right)^{2} + 7 \text{ x } 2^{x} - 1 = 0$

$\Rightarrow 8 \times (y)^{2} + 7 \text{ x } y - 1 = 0$ [Putting $2^{x} = y$]

$\Rightarrow 8y^{2} + 7y - 1 = 0$

$\Rightarrow 8y^{2} + 8y - 1y - 1 = 0$

$\Rightarrow 8y(y+1) - 1(y+1) = 0$

$\Rightarrow (y+1)(8y-1) = 0$

$\Rightarrow y + 1 = 0$ and $8y - 1 = 0$

$\Rightarrow 2^{x} + 1 = 0$ and $8 \times 2^{x} - 1 = 0$[Putting $y = 2^{x}$]

$\Rightarrow 2^{x} = -1$ and $8\text{x}2^{x} = 1$

$\Rightarrow 2^x = -1 \text{ and } 2^x = \frac{1}{8}$

$\Rightarrow 2^x = -1 \text{ and } 2^x = \frac{1}{2^3}$

$\Rightarrow 2^x = -1 \text{ and } 2^x = 2^{-3}$

In first case, value of x is not possible and in second case, value of x=-3

(ii) $\left(\sqrt{3}\right)^{x-3} = \left(\sqrt[4]{3}\right)^{x+1}$

$\Rightarrow \left(3^{\frac{1}{2}}\right)^{x-3} = \left(3^{\frac{1}{4}}\right)^{x+1} \Rightarrow 3^{\frac{1}{2}(x-3)} = 3^{\frac{1}{4}(x+1)}$

$\Rightarrow \frac{1}{2}(x-3) = \frac{1}{4}(x+1)$

[If bases are equal, powers are equal]

$\Rightarrow \frac{(x-3)}{2} = \frac{(x+1)}{4}$

$\Rightarrow 4(x-3) = 2(x+1) \qquad \text{[Cross– multplying]}$

$\Rightarrow 4x - 12 = 2x + 2 \Rightarrow 4x - 2x = 2 + 12$

$\Rightarrow 2x = 14 \Rightarrow x = \frac{14}{2} \Rightarrow x = 7$

8. (i) L.H.S. $\left(\frac{x^a}{x^b}\right)^{a+b-c}\left(\frac{x^b}{x^c}\right)^{b+c-a}\left(\frac{x^c}{x^a}\right)^{c+a-b} = 1$

$= \left(x^{a-b}\right)^{a+b-c}\left(x^{b-c}\right)^{b+c-a}\left(x^{c-a}\right)^{c+a-b}$

$= x^{a^2+ab-ca-ab-b^2+bc}\ x^{b^2+bc-ab-bc-c^2+ca}\ x^{c^2+ac-bc-ac-a^2+ab} = x^{a^2-b^2-ca+bc+b^2-c^2-ab+ca+c^2-a^2-bc+ab}$

$= x^0 = 1 = \text{R.H.S.}$

(ii) $\left(\frac{x^{a(b-c)}}{x^{b(a-c)}}\right) \div \left(\frac{x^b}{x^a}\right)^c = \frac{x^{ab-ac}}{x^{ab-bc}} \div (x^{b-a})^c = \frac{x^{ab-ac}}{x^{ab-bc}} \div x^{bc-ca} = x^{ab-ca-ab+bc-bc+ca} = x^0 = 1 = \text{R.H.S}$

9. Let $a^x = b^y = c^z = k$

$\therefore a = k^{\frac{1}{x}}, b = k^{\frac{1}{y}} \text{ and } c = k^{\frac{1}{z}}$

Now $b^2 = ac$

$\Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}} \times k^{\frac{1}{z}} \Rightarrow k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}}$

Comparing, we get,

$\frac{2}{y} = \frac{1}{x} + \frac{1}{z} \Rightarrow \frac{2}{y} = \frac{x+z}{xz} \Rightarrow y(x+z) = 2xz$

$$y=\frac{2xz}{x+z}$$

10. Let $5^{-p}=4^{-q}=20^{r}=k$, then

$$5^{-p}=k\Rightarrow 5=k^{\frac{1}{-p}}$$

$$4^{-q}=k\Rightarrow 4=k^{\frac{1}{-q}}$$

$$20^{r}=k\Rightarrow 20=k^{\frac{1}{r}}$$

$$20=4x5=\left(k^{\frac{1}{-q}}xk^{\frac{1}{-p}}\right)=k^{\frac{1}{r}} \qquad \Rightarrow k^{\frac{1}{-q}+\frac{1}{-p}}=k^{\frac{1}{r}}$$

Comparing, we get

$$\frac{-1}{p}-\frac{1}{q}=\frac{1}{r}\Rightarrow\frac{-1}{p}-\frac{1}{q}-\frac{1}{r}=0 \qquad \Rightarrow\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=0$$

11. $$\frac{3^{n+1}}{3^{n(n-1)}}\div\frac{9^{n+1}}{\left(3^{n+1}\right)^{(n-1)}}$$

$$\frac{3^{n+1}}{3^{n(n-1)}}\times\frac{\left(3^{n+1}\right)^{(n-1)}}{9^{n+1}}=\frac{3^{n+1}}{3^{n^2-n}}\times\frac{3^{(n+1)(n-1)}}{(3x3)^{n+1}}$$

$$\frac{3^{n+1}}{3^{n^2-n}}\times\frac{3^{(n^2)-(1)^2}}{(3^2)^{n+1}}=\frac{3^{n+1}}{3^{n^2-n}}\times\frac{3^{n^2-1}}{3^{2(n+1)}}$$

$$\frac{3^{n+1}}{3^{n^2-n}}\times\frac{3^{n^2-1}}{3^{2n+2}}=3^{n+1+n^2-1-(n^2-n)-(2n+2)}$$

$$3^{n+1+n^2-1-n^2+n-2n-2}=3^{-2}=\frac{1}{3^2}=\frac{1}{3x3}=\frac{1}{9}$$

12. $$\log\frac{75}{16}-2\log\frac{5}{9}+\log\frac{32}{243}=\log\frac{75}{16}-\log\left(\frac{5}{9}\right)^2+\log\frac{32}{243} \;\; .... \left[n\log_a m=\log_a m^n\right]$$

$$=\log\frac{75}{16}-\log\frac{25}{81}+\log\frac{32}{243}$$

$$=\log\frac{\frac{75}{16}}{\frac{25}{81}}+\log\frac{32}{243} \;\; \cdot \left[\log_a\frac{m}{n}=\log_a m-\log_a n\right]$$

$= \log \frac{75}{16} \times \frac{81}{25} + \log \frac{32}{243}$

$= \log \frac{243}{16} + \log \frac{32}{243}$

$= \log \frac{243}{16} \times \frac{32}{243}$ $\quad [\log_a mn = \log_a m + \log_a n]$

$= \log 1 \times 2 = \log 2$

13. (i) $\log V = 2 \log 2 - \log 3 + \log \pi + 3 \log r$

$\Rightarrow \log V = \log 2^2 - \log 3 + \log \pi + \log r^3$

$\Rightarrow \log V = \log 4 + \log \pi + \log r^3 - \log 3$

$\Rightarrow \log V = \log (4 \times \pi \times r^3) - \log 3$

$\Rightarrow \log V = \log \frac{4\pi r^3}{3}$

$\Rightarrow V = \frac{4\pi r^3}{3}$

(ii) $\log F = \log G + \log m_1 + \log m_2 - 2 \log d$

$\Rightarrow \log F = \log (G \times m_1 \times m_2) - \log d^2$

$\Rightarrow \log F = \log \frac{Gm_1m_2}{d^2}$

$\Rightarrow F = \frac{Gm_1m_2}{d^2}$

$\Rightarrow F = G\frac{m_1m_2}{d^2}$

14. (i) $\log_{10} 8 + \log_{10} 25 + 2 \log_{10} 4 - \log_{10} 32$

$= \log_{10} 8 + \log_{10} 25 + \log_{10} 4^2 - \log_{10} 32$ $\quad [\log_a m^n = n \log_a m]$

$= \log_{10} 8 + \log_{10} 25 + \log_{10} 16 - \log_{10} 32$

$= \log_{10} 8 \text{ x } 25 \text{ x } 16 - \log_{10} 32$ $\quad [\log_a lmn = \log_a l + \log_a m + \log_a n]$

$= \log_{10} 3200 - \log_{10} 32$

$= \log_{10}\frac{3200}{32}\left[\log_a\frac{m}{n} = \log_a m = \log_a n\right]$

$= \log_{10}100 = \log_{10}10^2 = 2\log_{10}10 = 2$

(ii) $\log 4 + \frac{1}{3}\log 125 - \frac{1}{5}\log 32 = \log 4 + \log(125)^{\frac{1}{3}} - \log(32)^{\frac{1}{5}}$. $\left[\log_a m^n = n\log_a m\right]$

$= \log 4 + \log 5 - \log 2$

$= \log(4\times 5) - \log 2$. $\left[\log_a mn = \log_a m + \log_a n\right]$

$= \log 20 - \log 2$

$= \log\frac{20}{2}.\left[\log_a\frac{m}{n} = \log_a m - \log_a n\right]$

$= \log 10 = 1$

15. L.H.S. : $2\log\frac{15}{18} - \log\frac{25}{162} + \log\frac{4}{9}$

$= \log\left(\frac{15}{18}\right)^2 - \log\frac{25}{162} + \log\frac{4}{9}$ $\left[\log_a m^n = n\log_a m\right]$

$= \log\frac{15}{18}\times\frac{15}{18} + \log\frac{4}{9} - \log\frac{25}{162}$

$= \log\frac{15}{18}\times\frac{15}{18}\times\frac{4}{9} - \log\frac{25}{162}$

$= \log\frac{\frac{15}{18}\times\frac{15}{18}\times\frac{4}{9}}{\frac{25}{162}}$ $\left[\log_a\frac{m}{n} = \log_a m - \log_a n\right]$

$= \log\frac{15}{18}\times\frac{15}{18}\times\frac{4}{9}\times\frac{162}{25} = \log\frac{3}{18}\times\frac{3}{18}\times\frac{4}{1}\times\frac{18}{1}$

$= \log\frac{1}{6}\times\frac{1}{6}\times 72 = \log\frac{72}{36} = \log 2 =$ R.H.S

16. $x - \log 48 + 3\log 2 = \frac{1}{3}\log 125 - \log 3$

$\Rightarrow x = \frac{1}{3}\log 125 - \log 3 + \log 48 - 3\log 2$

$\Rightarrow x = \log(125)^{\frac{1}{3}} - \log 3 + \log 48 - \log 2^3$

$\left[\log_a m^n = n\log_a m\right]$

$\Rightarrow x = \log 5 - \log 3 + \log 48 - \log 8$

$\Rightarrow x = (\log 5 + \log 48) - (\log 3 + \log 8)$

$\Rightarrow x = \log 5 \times 48 - \log 3 \times 8$

$[\log_a mn = \log_a m + \log_a n]$

$\Rightarrow x = \log \frac{5 \times 48}{3 \times 8}\left[\log_a \frac{m}{n} = \log_a m - \log_a n\right]$

$\Rightarrow x = \log \frac{5 \times 6}{3 \times 1} \Rightarrow x = \log 5 \times 2 \Rightarrow x = \log 10$

$\Rightarrow x = 1 \qquad [\because \log 10 = 1]$

17. $\log(x+4) - \log(x-4) = 1$

$\Rightarrow \log_{10}(x+4) - \log_{10}(x-4) = 1$

$\Rightarrow \log_{10}\frac{(x+4)}{(x-4)} = 1 \quad \left[\log_a m - \log_a n = \log_a \frac{m}{n}\right]$

$\Rightarrow \log_{10}\frac{(x+4)}{(x-4)} = 1 = \log_{10} 10$

$\Rightarrow \frac{(x+4)}{(x-4)} = 10$

$\Rightarrow (x+4) = 10x - 40$

$\Rightarrow x - 10x = -40 - 4$

$\Rightarrow -9x = -44$

$\Rightarrow 9x = 44 \Rightarrow x = \frac{44}{9}$

18. (i) $\log 12 = \log 2 \times 2 \times 3 = \log 2^2 \times 3$

$= \log 2^2 + \log 3 [\log_a m^n = \log_a m + \log_a n]$

$= 2\log 2 + \log 3 [\log_a m^n = n \log_a m]$

$= 2\log_{10} 2 + \log_{10} 3$

$= 2x + y$ [Putting $\log_{10} 2 = x, \log_{10} 3 = y$]

$= 2x + y$

(ii) $\log 2.25 = \log \frac{225}{100} \Rightarrow \log \frac{25x9}{25x4}$

$\Rightarrow \log \frac{9}{4} \Rightarrow \log\left(\frac{3}{2}\right)^2 = 2\log \frac{3}{2} \Rightarrow 2[\log 3 - \log 2] = 2[\log_{10} 3 - \log_{10} 2] = 2[y - x]$

[Putting $\log_{10} 3 = y, \log_{10} 2 = x$]

$= 2y - 2x$

(iii) $\log 3\frac{1}{8} = \log \frac{25}{8} = \log_{10} \frac{25}{8} x \frac{4}{4}$

$= \log_{10}\frac{100}{32} = \log_{10}\frac{100}{2\times2\times2\times2\times2} = \log_{10}\frac{100}{2^5} = \log_{10}100 - \log_{10}2^5$

$\left[\log_a\frac{m}{n} = \log_a m - \log_a n\right]$

$= 2 - 5\log_{10}{}^{2}$

$= 2 - 5x\left[\text{Putting } \log_{10}2 = x\right]$

19. log (a + b) = log a + log b

$\Rightarrow \log(a+b) = \log ab$

$[\log_a mn = \log_a m + \log_a n]$

$\Rightarrow a + b = ab \Rightarrow a - ab = -b$

$\Rightarrow -ab + a = -b \Rightarrow -a(b-1) = -b$

$\Rightarrow a(b-1) = b \Rightarrow a = \frac{b}{b-1}$

20. $l = \log\frac{a^2}{bc}, m = \log\frac{b^2}{ca}$ and $n = \log\frac{c^2}{ab}$

$l + m + n = \log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab}$

$= \log\frac{a^2}{bc}\times\frac{b^2}{ca}\times\frac{c^2}{ab}$ $\left[\log_a lmn = \log_a l + \log_a m + \log_a n\right]$

$= \log\frac{a^2b^2c^2}{a^2b^2c^2} \Rightarrow \log 1 \Rightarrow \log 1\times\frac{10}{10} \Rightarrow \log\frac{10}{10}$

$= \log10 - \log10$ $\left[\log_a\frac{m}{n} = \log_a m - \log_a n\right]$

$= 1 - 1 = 0$

21. $\log\frac{x-y}{2} = \frac{1}{2}\left(\log x + \log y\right) = \frac{1}{2}\left(\log xy\right) = \log(xy)^{1/2}$

$\therefore \frac{x-y}{2} = (xy)^{1/2}$

Squaring both sides

$\left(\frac{x-y}{2}\right)^2 = \left[(xy)^{\frac{1}{2}}\right]^2$

$\Rightarrow \frac{x^2+y^2-2xy}{4} = xy$

$\Rightarrow x^2 + y^2 - 2xy = 4xy$

$\Rightarrow x^2 + y^2 = 4xy + 2xy$

$\Rightarrow x^2 + y^2 = 6xy$

22. $a^2 + b^2 = 23ab$

$a^2 + b^2 + 2ab = 23ab + 2ab$

(Adding 2ab both sides)

$a^2 + b^2 + 2ab = 25ab$

$\Rightarrow (a+b)^2 = 25ab$

$\Rightarrow \frac{(a+b)^2}{25} = ab$

$\Rightarrow \left(\frac{a+b}{5}\right)^2 = ab$

Taking log both sides,

$\log\left(\frac{a+b}{5}\right)^2 = \log ab$

$\Rightarrow 2\log\frac{a+b}{5} = \log ab$

$\Rightarrow \log\frac{a+b}{5} = \frac{1}{2}\log ab$

$\Rightarrow \log\frac{a+b}{5} = \frac{1}{2}(\log a + \log b)$

23. $a > 0$, $\log ab = \log\frac{a}{b} + 2\log 2 = 2$

(i)$\log ab = 2 = 2\times 1 = 2\log 10 \ = \log 10^2 = \log 100$

$\therefore ab = 100$ ....(i)

(ii)$\log\frac{a}{b} + 2\log 2 = 2$

$\log\frac{a}{b} + \log 2^2 = \log 100$

$(\because \log 100 = 2)$

$\log\frac{a}{b} + \log 2^2 = \log 100$

$\Rightarrow \log\frac{4a}{b} = \log 100$

$\frac{4a}{b} = 100$

$4a = 100b \Rightarrow a = 25b$ ....(ii)

From (i) $25b.b = 100$

$\Rightarrow 25b^2 = 100 \Rightarrow b^2 = 4 = (+2)^2$

$\Rightarrow b = 2$ $\qquad (\because b > 0)$

$\therefore a \times 2 = 100 \Rightarrow a = \frac{100}{2} = 50$

Hence $a = 50, b = 2$

24. $\log_2(x+y) = \log_3(x-y) = \frac{\log 25}{\log 0.2}$

Now $\frac{\log 25}{\log 0.2} = \frac{\log 5^2}{\log 5^{-1}} = \frac{2\log 5}{\log 5^{-1}} = \frac{2}{-1} = -2$

Now $\log_2(a+b) = -2$

$\Rightarrow 2^{-2} = a + b \Rightarrow a + b = \frac{1}{2^2} = \frac{1}{4}$ ....(i)

and $\log_3(a-b) = -2 \Rightarrow 3^{-2} = a - b$

$\Rightarrow a - b = \frac{1}{3^2} = \frac{1}{9}$ ....(ii)

Adding (i) and (ii)

$2a = \frac{1}{4} + \frac{1}{9} = \frac{9+4}{36} = \frac{13}{36}$

$\therefore a = \frac{13}{36x2} = \frac{13}{72}$

and subtracting (ii) from (i), we get

$2b = \frac{1}{4} - \frac{1}{9} = \frac{9-4}{36} = \frac{5}{36}$

$b = \frac{5}{36 \times 2} = \frac{5}{72}$

$\therefore a = \frac{13}{72}$ and $b = \frac{5}{72}$