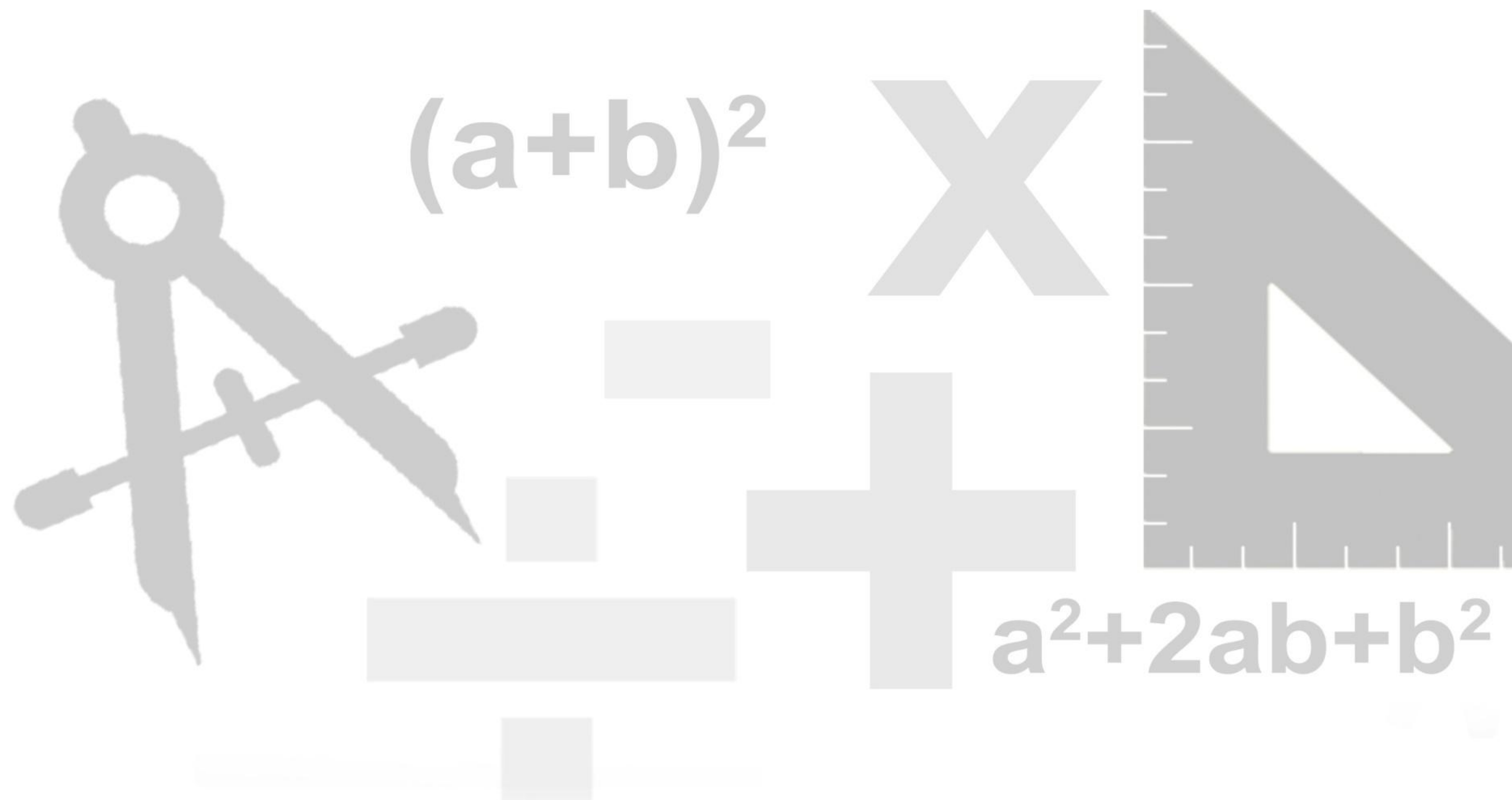
MATHS



Integrals

Top Concepts

1. Integration is the inverse process of differentiation. The process of finding the function from its primitive is known as integration or antidifferentiation.

2. The problem of finding a function whenever its derivative is given leads to indefinite form of integrals.

3. The problem of finding the area bounded by the graph of a function under certain conditions leads to a definite form of integrals.

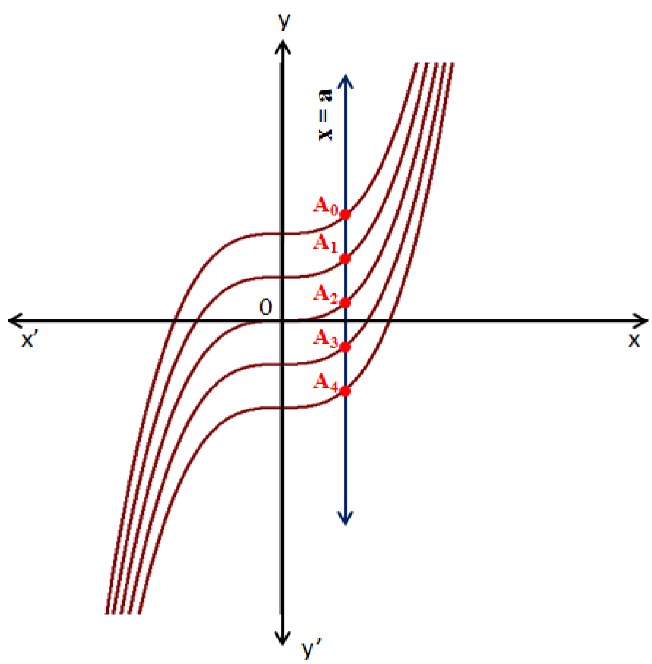
4. Indefinite and definite integrals together constitute **Integral Calculus**.

5. Indefinite integral $\int f(x)dx = F(x) + C$, where F(x) is the antiderivative of f(x).

6. Functions with same derivatives differ by a constant.

7. $\int f(x)dx$ means integral of f with respect to x, f(x) is the integrand, x is the variable of integration and C is the constant of integration.

8. Geometrically indefinite integral is the collection of family of curves, each of which can be obtained by translating one of the curves parallel to itself.
   Family of curves representing the integral of $3x^2$



$\int f(x)dx = F(x) + C$ represents a family of curves where different values of C correspond to different members of the family, and these members are obtained by shifting any one of the curves parallel to itself.

9. **Properties of antiderivatives**

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

$\int kf(x)dx = k\int f(x)dx$ for any real number k

$$\int [k_1f_1(x)+k_2f_2(x)+\ldots\ldots+k_nf_n(x)]dx = k_1\int f_1(x)dx + k_2\int f_2(x)dx + \ldots. + k_n\int f_n(x)dx$$

where $k_1, k_2, \ldots k_n$ are real numbers and $f_1, f_2, \ldots f_n$ are real functions.

10. Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

11. **Comparison between differentiation and integration**
    1. Both are operations on functions.
    2. Both satisfy the property of linearity.
    3. All functions are not differentiable and all functions are not integrable.
    4. The derivative of a function is a unique function, but the integral of a function is not.
    5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
    6. The derivative is defined at a point P and the integral of a function is defined over an interval.
    7. Geometrical meaning: The derivative of a function represents the slope of the tangent to the corresponding curve at a point. The indefinite integral of a function represents a family of curves placed parallel to each other having parallel tangents at the points of intersection of the family with the lines perpendicular to the axis.
    8. The derivative is used for finding some physical quantities such as the velocity of a moving particle when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time t is known.
    9. Differentiation and integration, both are processes involving limits.
    10. By knowing one antiderivative of function f, an infinite number of antiderivatives can be obtained.

11. Integration can be done by using many methods. Prominent among them are
    i. Integration by substitution
    ii. Integration using partial fractions
    iii. Integration by parts
    iv. Integration using trigonometric identities

12. A change in the variable of integration often reduces an integral to one of the fundamental integrals. Some standard substitutions are

    $x^2 + a^2$; substitute $x = a \tan \theta$

    $\sqrt{x^2-a^2}$; substitute $x = a \sec \theta$

    $\sqrt{a^2-x^2}$; substitute $x = a \sin \theta$ or $a \cos \theta$

13. A function of the form $\frac{P(x)}{Q(x)}$ is known as a rational function. Rational functions can be integrated using partial fractions.

14. **Partial fraction decomposition** or **partial fraction expansion** is used to reduce the degree of either the numerator or the denominator of a rational function.

15. **Integration using partial fractions**

A rational function $\frac{P(x)}{Q(x)}$ can be expressed as the sum of partial fractions if $\frac{P(x)}{Q(x)}$. This takes any of the forms:

- $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$, $a \neq b$
- $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$
- $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
- $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
- $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further.

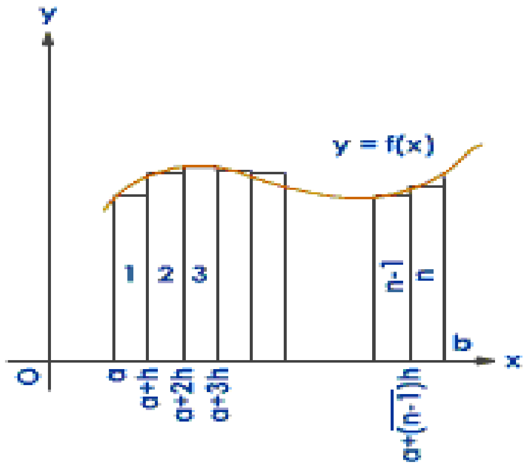
16. To find the integral of the product of two functions, integration by parts is used.
I and II functions are chosen using the ILATE rule:
I - inverse trigonometric
L - logarithmic
A - algebraic
T - trigonometric
E - exponential is used to identify the first function.

17. **Integration by parts**

Integral of the product of two functions = (first function) × (integral of the second function) – integral of [(differential coefficient of the first function) × (integral of the second function)]

$$\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$$, where f_1 and f_2 are functions of x.

18. Definite integral $\int_a^b f(x)dx$ of the function f(x) from limits a to b represents the area enclosed by the graph of the function f(x), the x-axis and the vertical markers x = ‘a’ and x = ‘b’.



19. **Definite integral as the limit of a sum:** The process of evaluating a definite integral by using the definition is called integration as the limit of a sum or integration from first principles.

20. Method of evaluating $\int_a^b f(x)dx$

    (i) Calculate antiderivative F(x)
    (ii) Calculate F(b) – F(a)

21. Area function

    $A(x) = \int_a^x f(x)dx$, if x is a point in [a, b].

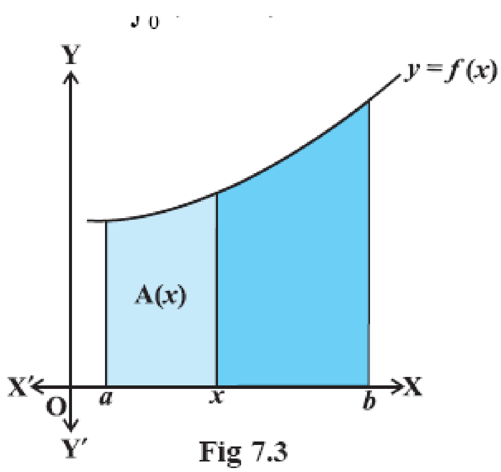


Fig 7.3

22. **Fundamental Theorem of Integral Calculus**

- **First fundamental theorem** of integral calculus: If area function, $A(x) = \int_a^x f(x)dx$ for all $x \geq a$, and f is continuous on [a, b]. Then $A'(x) = f(x)$ for all $x \in [a, b]$
- **Second fundamental theorem** of integral calculus: Let *f* be a continuous function of x in the closed interval [*a*, *b*] and let F be antiderivative of $\frac{d}{dx}F(x) = f(x)$ for all x in domain of f, then

$$\int_a^b f(x)dx = [F(x) + C]_a^b = F(b) - F(a)$$

Top Formulae

1. Some Standard Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
- $\int dx = x + C$
- $\int \cos x\, dx = \sin x + C$
- $\int \sin x\, dx = -\cos x + C$
- $\int \sec^2 x\, dx = \tan x + C$
- $\int \operatorname{cosec}^2 x\, dx = -\cot x + C$
- $\int \sec x \tan x\, dx = \sec x + C$
- $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$
- $\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
- $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$
- $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$
- $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\log a} + C$

- $\int \frac{1}{x} dx = \log|x| + C$
- $\int \tan x \, dx = \log|\sec x| + C$
- $\int \cot x \, dx = \log|\sin x| + C$
- $\int \sec x \, dx = \log|\sec x + \tan x| + C$
- $\int \operatorname{co}\sec x \, dx = \log|\operatorname{co}\sec x - \cot x| + C$
- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$
- $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C$
- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
- $\int a^{bx+c} dx = \frac{1}{b} \cdot \frac{a^{bx+c}}{\log a} + C, a > 0, a \neq 1$
- $\int \sin(ax+b) dx = -\frac{1}{a}\cos(ax+b) + C$
- $\int \cos(ax+b) dx = \frac{1}{a}\sin(ax+b) + C$
- $\int \tan(ax+b) dx = \frac{1}{a}\log|\sec(ax+b)| + C$
- $\int \cot(ax+b) dx = \frac{1}{a}\log|\sin(ax+b)| + C$
- $\int \sec^2(ax+b) dx = \frac{1}{a}\tan(ax+b) + C$
- $\int \operatorname{co}\sec^2(ax+b) dx = -\frac{1}{a}\cot(ax+b) + C$
- $\int \sec(ax+b)\tan(ax+b) dx = \frac{1}{a}\sec(ax+b) + C$
- $\int \operatorname{co}\sec(ax+b)\cot(ax+b) dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$
- $\int \sec(ax+b) dx = \frac{1}{a}\log|\sec(ax+b) + \tan(ax+b)| + C$
- $\int \operatorname{co}\sec(ax+b) dx = \frac{1}{a}\log|\operatorname{co}\sec(ax+b) - \cot(ax+b)| + C$

2. Integral of some special functions

- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
- $\int -\frac{dx}{x^2 + a^2} = \frac{1}{a}\cot^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$
- $\int -\frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$
- $\int -\frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\text{cosec}^{-1}\frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + C$
- $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + C$
- $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2\sin^{-1}\frac{x}{a} + C$
- $\int \sqrt{a^2 + x^2}\,dx = \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{1}{2}a^2\log\left|x + \sqrt{a^2 + x^2}\right| + C$
- $\int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2\log\left|x + \sqrt{x^2 - a^2}\right| + C$

3. Integration by parts

(i) $\int f_1(x).f_2(x)dx = f_1(x)\int f_2(x)dx - \int\left[\frac{d}{dx}f_1(x).\int f_2(x)dx\right]dx$, where f_1 and f_2 are functions of x.

(ii) $\int e^x\left(f(x)+f'(x)\right)dx = e^x f(x)+C$

4. Integral as the limit of sums:

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + ... + f(a+(n-1)h)\right] \text{ where } h = \frac{b-a}{n}$$

5. Different methods of integration

1. Evaluation of integrals of the form $\frac{p(x)}{(ax+b)^n}, n \in N,$ where $p(x)$ is a polynomial

   Steps:

   i. Check whether degree of $p(x) \geq$ or $\leq n$

   ii. If degree of p(x) < n, then express p(x) in the form
   $A_0 + A_1(ax+b) + A_2(ax+b)^2 + \ldots + A_{n-1}(ax+b)^{n-1}$

   iii. Write $\frac{p(x)}{(ax+b)^n}$ as $\frac{A_0}{(ax+b)^n} + \frac{A_1}{(ax+b)^{n-1}} + \frac{A_2}{(ax+b)^{n-2}} + \ldots + \frac{A_{n-1}}{(ax+b)}$

   iv. Evaluate
   $$\int \frac{p(x)}{(ax+b)^n}dx = A_0\int \frac{1}{(ax+b)^n}dx + A_1\int \frac{1}{(ax+b)^{n-1}}dx$$
   $$+A_2\int \frac{1}{(ax+b)^{n-2}}dx + \ldots + A_{n-1}\int \frac{1}{(ax+b)}dx$$

   v. If degree of $p(x) \geq n$, then divide p(x) by $(ax+b)^n$ and express
   $\frac{p(x)}{(ax+b)^n}$ as $q(x) + \frac{r(x)}{(ax+b)^n}$, where degree of
   $r(x)$ is less than n

   vi. Use steps (ii) and (iii) to evaluate $\int \frac{r(x)}{(ax+b)^n}dx$

2. Evaluation of integrals of the form $\int (ax+b)\sqrt{cx+d}dx$

   Steps:

   i. Represent $(ax+b)$ in terms of $(cx+d)$ as follows:
   $$(ax+b) = A(cx+d) + B$$

   ii. Find A and B by equating coefficients of like powers of x on both sides

   iii. Replace $(ax+b)$ by $A(cx+d) + B$ in the given integral to obtain
   $$\int (ax+b)\sqrt{cx+d}dx = \int |A(cx+d) + B|\sqrt{cx+d}dx$$
   $$= A\int (cx+d)^{\frac{3}{2}}dx + B\int \sqrt{cx+d}dx$$
   $$= \frac{2A}{5c}(cx+d)^{\frac{5}{2}} + \frac{2B}{5c}(cx+d)^{\frac{3}{2}} + C$$

3. Evaluation of integrals of the form $\int \frac{(ax+b)}{\sqrt{cx+d}}dx$

Steps:

i. Represent $(ax+b)$ in terms of $(cx+d)$ as follows:

$(ax+b) = A(cx+d)+B$

ii. Find A and B by equating coefficients of like powers of x on both sides

iii. Replace $(ax+b)$ by $A(cx+d)+B$ in the given integral to obtain

$$\int \frac{(ax+b)}{\sqrt{cx+d}}dx = \int \frac{|A(cx+d)+B|}{\sqrt{cx+d}}dx$$

$$= A\int \sqrt{cx+d}\,dx + B\int \frac{1}{\sqrt{cx+d}}dx$$

$$= \frac{2A}{3c}(cx+d)^{\frac{3}{2}} + \frac{2B}{c}(cx+d)^{\frac{1}{2}} + C$$

4. Evaluation of integrals of the form $\int \sin^m x\,dx, \int \cos^m x\,dx,$ where $m \le 4$

Let us express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using the following identities:

i. $\sin^2 x = \frac{1-\cos 2x}{2}$

ii. $\cos^2 x = \frac{1+\cos 2x}{2}$

iii. $\sin 3x = 3\sin x - 4\sin^3 x$

iv. $\cos 3x = 4\cos^3 x - 3\cos x$

5. Evaluation of integrals of the form

$\int \sin mx \cdot \cos nx\,dx, \int \sin mx \cdot \sin nx\,dx, \int \cos mx \cdot \cos nx\,dx$

Let us use the following identities:

i. $2\sin A\cos B = \sin(A+B)+\sin(A-B)$

ii. $2\cos A\cos B = \cos(A+B)+\cos(A-B)$

iii. $2\cos A\sin B = \sin(A+B)-\sin(A-B)$

iv. $2\sin A\sin B = \cos(A-B)-\cos(A+B)$

6. Evaluation of integrals of the form $\int \frac{f'(x)}{f(x)}dx$

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + C$$

7. Evaluation of integrals of the form

$$\int (ax+b)^n p(x)\,dx, \int \frac{p(x)}{(ax+b)^n}dx, \text{ where } p(x) \text{ is a polynomial}$$

and n is a positive rational number

Steps:

i. Substitute $ax+b=v$ or $x=\frac{v-b}{a}$ and $dx=\frac{1}{a}dv$

ii. Now integrate with respect to v by using $\int v^n dv = \frac{v^{n+1}}{n+1}+C$

iii. Replace v by ax+b

8. Evaluation of integrals of the form

$$\int \tan^m x \cdot \sec^{2n} x dx, \int \cot^m x \cdot \operatorname{co}\sec^{2n} x dx, m, n \in N$$

Steps:

i. Rewrite the given integral as $I = \int \tan^m x \cdot (\sec^2 x)^{(n-1)} \sec^2 x dx$

ii. Substitute $\tan x = v$ and $\sec^2 x dx = dv$

Therefore,

$$I = \int \tan^m x \cdot (\sec^2 x)^{(n-1)} \sec^2 x dx$$

$$= \int \tan^m x \cdot (1+\tan^2 x)^{(n-1)} \sec^2 x dx$$

$$= \int v^m \cdot (1+v^2)^{(n-1)} dv$$

iii. Use the binomial theorem to expand $(1+v^2)^{(n-1)}$ in step (ii) and integrate

iv. Replace v by tanx in step (iii)

9. Evaluation of integrals of the form

$$\int \tan^{2m+1} x \cdot \sec^{2n+1} x dx, \int \cot^m x \cdot \operatorname{co}\sec^{2n} x dx, \text{ where}$$

m and n are nonnegative integers

Steps:

i. Rewrite the given integral as $I = \int (\tan^2 x)^m \cdot (\sec x)^{2n} \sec x \cdot \tan x dx$

ii. Put $\sec x = v$ and $\sec x \tan x dx = dv$

Therefore,

$$I = \int (\sec^2 x - 1)^m \cdot (\sec x)^{2n} \sec x \cdot \tan x dx$$

$$= \int (v^2-1)^m v^n dv$$

iii. Use the binomial theorem to expand $(v^2-1)^m$ in step (ii) and integrate

iv. Replace v by secx in step (iii)

10. Evaluation of integrals of the form $\int \sin^m x \cdot \cos^n x dx,\ \text{where } m,n \in N$

Steps:

i. Check the exponents of sinx and cosx

ii. If the exponent of sinx is an odd positive integer, then put cosx = v

If the exponent of cosx is an odd positive integer, then put sinx = v

If the exponents of both sinx and cosx are odd positive integers, then put either sinx = v or cosx = v

If the exponents of both sinx and cosx are even positive integers, then rewrite $\sin^m x \cos^n x$ in terms of sines and cosines of multiples of x by using trigonometric results

iii. Evaluate the integral in step (ii)

11. Evaluation of integrals of the form

$\int \sin^m x \cdot \cos^n x dx,\ \text{where } m,n \in Q,$ such that

m+n is a negative even integer.

Steps:

i. Represent the integrand in terms of tanx and $\sec^2 x$ by dividing the numerator and denominator by $\cos^k x$, where k = −(m + n)

ii. Put tanx = v

12. Evaluation of integrals of the form $\int \frac{dx}{ax^2 + bx + c}$

Steps:

i. Multiply and divide the integrand by x^2 and make the coefficient of x^2 unity

ii. Observe the coefficient of x

iii. Add and subtract $\left(\frac{1}{2}\text{coefficient } x\right)^2$ to the expression in the denominator

iv. Express the expression in the denominator in the form $\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$

v. Use the appropriate formula to integrate

13. Evaluation of integrals of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$

Steps:

i. Multiply and divide the integrand by x^2 and make the coefficient of x^2 unity

ii. Observe the coefficient of x

iii. Add and subtract $\left(\frac{1}{2}\text{coefficient } x\right)^2$ inside the square root

iv. Express the expression inside the square root in the form $\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$

v. Use the appropriate formula to integrate

14. Evaluation of integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$

Steps:

i. Rewrite the numerator as follows:

$$px+q=A\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+B$$

$$\Rightarrow px+q=A\{2ax+b\}+B$$

ii. Find the values of A and B by equating the coefficients of like powers of x on both sides

iii. Substitute px+q by $A\{2ax+b\}+B$ in the given integral

Therefore, $\int \frac{px+q}{ax^2+bx+c}dx=A\int \frac{2ax+b}{ax^2+bx+c}dx+B\int \frac{1}{ax^2+bx+c}dx$

iv. Integrate the right-hand side and substitute the values of A and B

15. Evaluation of integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$, where p(x) is a polynomial degree greater than or equal to 2

Steps:

i. Divide the numerator by the denominator, and rewrite the integrand as

$q(x)+\frac{r(x)}{ax^2+bx+c}$, where r(x) is a linear function of x

ii. Thus,

$$\int \frac{px+q}{ax^2+bx+c}dx=\int q(x)dx+\int \frac{r(x)}{ax^2+bx+c}dx$$

iii. Integrate the second integral on the right-hand side and apply the appropriate method

16. Evaluation of integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Steps:

i. Rewrite the numerator as follows:

$$px+q=A\left\{\frac{d}{dx}\left(ax^2+bx+c\right)\right\}+B$$

$$\Rightarrow px+q=A\{2ax+b\}+B$$

ii. Find the values of A and B by equating the coefficients of like powers of x on both sides

iii. Substitute px + q by $A\{2ax+b\}+B$ in the given integral

Therefore, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx=A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx+B\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

iv. Integrate the right-hand side and substitute the values of A and B

17. Evaluation of integrals of the form

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x}, \int \frac{dx}{a + b\sin^2 x}, \int \frac{dx}{a + b\cos^2 x},$$

$$\int \frac{dx}{(a\sin x + b\cos x)^2}, \int \frac{dx}{a + b\sin^2 x + c\cos^2 x}$$

Steps:

i. Divide the numerator and denominator by $\cos^2 x$

ii. In the denominator, replace $\sec^2 x$ by $1 + \tan^2 x$

iii. Substitute $\tan x = v; \sec^2 x dx = dv$

iv. Apply the appropriate method to integrate the integral $\int \frac{dv}{av^2 + bv + c}$

18. Evaluation of integrals of the form

$$\int \frac{dx}{a\sin x + b\cos x}, \int \frac{dx}{a + b\sin x}, \int \frac{dx}{a + b\cos x},$$

$$\int \frac{dx}{a + b\sin x + c\cos x}$$

Steps:

i. Substitute $\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}, \cos x = \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

ii. In the numerator, replace $1 + \tan^2\frac{x}{2}$ by $\sec^2\frac{x}{2}$

iii. Substitute $\tan\frac{x}{2} = v; \frac{1}{2}\sec^2\frac{x}{2}dx = dv$

iv. Apply the appropriate method to integrate the integral $\int \frac{dv}{av^2 + bv + c}$

19. **Alternate method:** Evaluation of integrals of the form

$$\int \frac{dx}{a\sin x + b\cos x}$$

Steps:

Substitute $a = r\cos\theta, b = r\sin\theta;$

where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\Rightarrow a\sin x + b\cos x = r\cos\theta\sin x + r\sin\theta\cos x = r\sin(x + \theta)$$

$$\therefore \int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2 + b^2}}\log\left|\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right| + C$$

20. Evaluation of integrals of the form

$$\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$$

Steps:

i. Substitute

Numeraror $=$ A(Differentiation of denominator) $+$ B(Denominator)

That is asinx+bcosx=$A(c\cos x - d\sin x) + B(c\sin x + d\cos x)$

ii. Compare the coefficients of sinx and cosx on both the sides and get the values of A and B

iii. Replace the integrand by $A(c\cos x - d\sin x) + B(c\sin x + d\cos x)$

iv. Hence, the value of the integral $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$ is

$$= A\log\left|c\sin x + d\cos x\right| + \mu x + C$$

21. Evaluation of integrals of the form

$$\int \frac{a\sin x + b\cos x + c}{m\sin x + n\cos x + p}\,dx$$

Steps:

i. Substitute

Numeraror $=$ A(Differentiation of denominator) $+$ B(Denominator) $+$ K

That is asinx+bcosx+c =$A(m\cos x - n\sin x) + B(m\sin x + n\cos x + p) + K$

ii. Compare the coefficients of sinx and cosx and constant terms on both the sides and get the values of A, B and K

iii. Replace the integrand by $A(m\cos x - n\sin x) + B(m\sin x + n\cos x + p) + K$

iv. Hence, the value of the integral $\int \frac{a\sin x + b\cos x + c}{m\sin x + n\cos x + p}\,dx$ is

$$= A\log\left|m\sin x + n\cos x + p\right| + Bx + p\int \frac{1}{m\sin x + n\cos x + p}dx$$

v. Evaluate the integral on the right-hand side by any appropriate method

22. Evaluation of integrals of the form

$$\int e^x\left[f(x) + f'(x)\right]\,dx$$

Steps:

i. Write the given integral as

$$\int e^x\left|f(x) + f'(x)\right|dx = \int e^x f(x)dx + \int e^x f'(x)dx$$

ii. Find the integration for the first term by parts

iii. (Cancel out the second integral with the second term obtained by integration by parts

iv. Thus, the above result holds true for e^{kx}

$$\int e^{kx}\left[f(x) + f'(x)\right]\,dx = e^{kx} + C$$

23. Evaluation of the integrals of the form $\int \sqrt{ax^2 + bx + c}dx$

Steps:

i. Take ‘a’ common inside the square root so as to get $x^2 + \frac{b}{a}x + \frac{c}{a}$

ii. Add and subtract the appropriate term to $x^2 + \frac{b}{a}x + \frac{c}{a}$ to get the term $\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$

iii. Now evaluate the integral using the appropriate formulae

24. Evaluation of the integrals of the form $\int (mx + n)\sqrt{ax^2 + bx + c}dx$

Steps:

i. Rewrite mx + n as $mx + n = A\frac{d}{dx}\left(ax^2 + bx + c\right) + B$

That is, $mx + n = A(2ax + b) + B$

ii. Equate the coefficients of x and constant terms on both sides to get the values of A and B

iii. Substitute $mx + n$ by $A(2ax + b) + B$

iv. Now evaluate the integral using the appropriate formulae

25. Evaluation of the integrals of the form

$\int \frac{x^2 + 1}{x^4 + kx^2 + 1}dx, \int \frac{x^2 - 1}{x^4 + kx^2 + 1}dx, \int \frac{1}{x^4 + kx^2 + 1}dx$ where $k \in R$

Steps:

i. Divide the numerator and the denominator by x^2

ii. Write the denominator of the integrand in the form of $\left(x + \frac{1}{x}\right)^2 \pm m^2$

iii. Write $d\left(x + \frac{1}{x}\right)$ or $d\left(x - \frac{1}{x}\right)$ or both in the numerator

iv. Put $x + \frac{1}{x} = v$ or $x - \frac{1}{x} = v$

v. Evaluate the integral using the appropriate formula

26. Evaluation of integration of irrational algebraic functions,

$\int \frac{f(x)}{(gx + h)\sqrt{mx + n}}dx$ where $g,h,m,n \in R$

Step: Put $mx + n = v^2$ to evaluate the integral

27. Evaluation of integration of irrational algebraic functions,

$$\int \frac{f(x)}{(rx^2+gx+h)\sqrt{mx+n}}dx \text{ where } r,g,h,m,n \in R$$

Step: Put $mx+n=v^2$ to evaluate the integral

28. Evaluation of integration of irrational algebraic functions,

$$\int \frac{1}{(gx+h)\sqrt{mx^2+nx+p}}dx \text{ where } g,h,m,n,p \in R$$

Step: Put $gx+h=\frac{1}{v}$ to evaluate the integral

29. Evaluation of integration of irrational algebraic functions,

$$\int \frac{1}{(gx^2+h)\sqrt{mx^2+n}}dx \text{ where } g,h,m,n \in R$$

Step:

i. Put $x=\frac{1}{v}$

Therefore, $I=\int \frac{-vdv}{(g+hv^2)\sqrt{m+nv^2}}$

Now substitute $m+nv^2=w^2$

6. Properties of definite integrals

- $\int_a^b f(x)dx = \int_a^b f(t)dt$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$

  In particular, $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

- $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$
- $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, \text{if } f(2a-x) = f(x)$

  $= 0 \qquad , \text{if } f(2a-x) = -f(x)$

$\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx, \text{if } f(-x) = f(x)$

$= 0 \qquad , \text{if } f(-x) = -f(x)$