Sets, Relations and Functions

FORMULAE

Properties :

1. $A \cup B = B \cup A$
2. $(A \cup B) \cup C = A \cup (B \cup C)$
3. $A \cup \emptyset = A$
4. $A \cup A = A$
5. $A \cup A' = U$
6. If $A \subset B$ then $A \cup B = B$
7. $U \cup A = U$
8. $A \subset (A \cup B), B \subset (A \cup B)$
9. $A \cap B = B \cap A$
10. $(A \cap B) \subset A, (A \cap B) \subset B$
11. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
12. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
13. $(A \cap B) \cap C = A \cap (B \cap C)$
14. $\emptyset \cap A = \emptyset$
15. $A \cap A = A$
16. $A \cap A' = \emptyset$
17. If $A \subset B$ then $A \cap B = A$
18. $U \cap A = A$
19. $(A')' = A$
20. $\emptyset' = U$ where U is the universal set
21. $U' = \emptyset$

Difference of sets :

1. $A - B$ is a subset of A and $B - A$ is a subset of B.
2. The sets $A - B, A \cap B, B - A$ are mutually disjoint sets. The intersection of any of these two sets is the null set.
3. $A - B = A \cap B'$
4. $B - A = A' \cap B$
5. $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$

If $n(A) = m$ then $n[P(A)] = 2^m$ where $n[P(A)]$ is the number of subsets of set A.

If $n(A) = m$ then $n[P(A)] = 2^m - 1$ where $n[P(A)]$ is the number of proper subsets of set A.

De Morgan's Laws :

For any two sets A and B

1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B)' = A' \cup B'$

Some Important Results :

For given sets A, B

1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. When A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$
3. $n(A \cap B') + n(A \cap B) = n(A)$

4. $n(A' \cap B) + n(A \cap B) = n(B)$
5. $n(A \cap B') + n(A \cap B) + n(A' \cap B) = n(A \cup B)$
   For any sets A, B, and C
6. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Important results on Cartesian Product of sets

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
3. $A \times (B - C) = (A \times B) - (A \times C)$
4. If A and B are any two non-empty sets and $A \times B = B \times A$ then $A = B$
5. If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$
6. If $A \subseteq B$, then $A \times C \subseteq (B \times C)$ for any set C.
7. If $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$
8. For any set A, B, C, D
   $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
9. If A and B are two non-empty sets and $A \cap B$ has n elements, then $A \times B$ and $B \times A$ have n^2 elements in common.

Various types of relations :

Let A be a non-empty set then a relation R on A is said to be

1. Reflexive if $(a, a) \in R$ for all $a \in A$ i. e. aRa for all $a \in A$
2. Symmetric if $(a, b) \in R$ then $(b, a) \in R$ for all $a, b \in A$ i. e. aRb then bRa for all $a, b \in A$
3. Transitive if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ for all $a, b, c \in A$
   i.e. aRb, bRc then aRc for all $a, b, c \in A$
4. A relation which is reflexive, symmetric and transitive is called an equaivalence relation.

Types of functions :

1. **One-one function :**
   A function $A \rightarrow B$ is said to be one-one if distinct elements in A have distinct images in B. i.e. $x_1, x_2 \in A$ such that $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$
2. **Onto Function :**
   A function $A \rightarrow B$ is said to be onto if every element of B is the image of some element of A under the function f. If f is onto then for every $y \in B$ their exists at least one element $x \in A$ such that $y = f(x)$. If f is onto then range of f = codomain of f
3. **Into function :**
   The function $A \rightarrow B$ is such that there exists at least one element in B which has no preimage in A, then f is said to be an into function.
   Hence, the range of f is a proper subset of its codomain B.
4. **Even and odd function :**
   A function f is said to be an even function, if $f(-x) = f(x)$ for all $x \in R$
   A function f is said to be an odd function, if $f(-x) = -f(x)$ for all $x \in R$
5. **The greatest integer function (or step function) :**
   If $x \in R$ then the greatest integer not exceeding x is denoted by [x].
6. **Composite function :**
   Let $f:A \rightarrow B$ and $g:B \rightarrow C$ be two functions where domain of g is same as co-domain of f then we define the composite function of f and g from A to C denoted by $g \circ f(x) = g[f(x)]$
7. **Inverse function :**
   If $f:A \rightarrow B$ is one-one onto, $g:B \rightarrow A$ which is also one-one, onto such that $g \circ f:A \rightarrow A$ and

$f \circ g : B \rightarrow B$ are both identity functions then f and g are called as inverse functions of each other. Function g is denoted by f^{-1} and is read as f-inverse. Hence, a function f^{-1} as $f^{-1} : B \rightarrow A$ such that if $f(x) = y$ then $f^{-1}(y) = x$.

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Complex Number

1. **The complex number system**
   Z = a + ib, then a – ib is called conjugate of z and is denoted by $\bar{z}$

2. **Equality In Complex Number** : $z_1 = z_2 \Rightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $I_m(z_1) = I_m(z_2)$

3. **Representation of A Complex Number :**

4. **Properties of arguments**
   i. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2m\pi$ for some integer m.
   ii. $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2) + 2m\pi$ for some integer m.
   iii. $\arg(z^2) = 2\arg(z) + 2m\pi$ for some integer m.
   iv. $\arg(z) = 0 \quad \Leftrightarrow \quad$ z is a positive real number
   v. $\arg(z) = \pm \pi/2 \quad \Leftrightarrow \quad$ z is purely imaginary and $z \neq 0$

5. **Properties of conjugate**
   i. $|z| = |\bar{z}|$
   ii. $z\bar{z} = |z|^2$
   iii. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
   iv. $z + \bar{z} = 2\,\text{Re}(z)$
   v. $z - \bar{z} = 2i\,\text{Im}(z)$
   vi. $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$
   vii. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)$
   viii. $z + \bar{z} = 0 \Rightarrow z = 0$ or $z = \text{Im}(z)$
   ix. $|z_1 \pm z_2|^2 = (z_1 \pm z_2)\,\overline{(z_1 \pm z_2)} = |z_1|^2 + |z_2|^2 \pm z_1\overline{z_2} \pm \overline{z_1}z_2$
   x. $\overline{(\bar{z})} = z$
   xi. if w = f(z), then $\overline{w} = f(\bar{z})$
   xii. $z = \bar{z} \Rightarrow z$ is purely imaginary
   xiii. $\arg(z) + \arg(\bar{z}) = 0$
   xiv. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

6. **Rotation theorem**

   if $P(z_1)$, $Q(z_2)$ and $R(z_3)$ are three complex numbers and $\angle PQR = \theta$, then $\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| e^{i\theta}$

7. **Demoivre's Theorem :**

Case I: if n is any integer then

i. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

ii. $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\,\ldots\ldots(\cos\theta_n + i\sin\theta_n) = \cos(\theta_1+\theta_2+\theta_3+\ldots\ldots\theta_n) + i\sin(\theta_1+\theta_2+\theta_3+\ldots\ldots+\theta_n)$

Case II: if $p,q \in Z$ and $q \neq 0$ then $(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\dfrac{2k\pi+p\theta}{q}\right) + i\sin\left(\dfrac{2k\pi+p\theta}{q}\right)$

Where k = 0, 1, 2, 3,……………, q – 1

8. **Cube Root of Unity :**

i. The cube roots of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

ii. if ω is one of the imaginary cube roots of unity then $1+\omega+\omega^2 = 0$. In general $1+\omega^r+\omega^{2r} = 0$; where $r \in I$ but is not the multiple of 3.

9. **Logarithm of a Complex Quantity:**

(i) $\text{Log}_e(\alpha+i\beta) = \dfrac{1}{2}\text{Log}_e(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}\right)$ where $n \in I$.

10. **Geometrical Properties:**

Distance formula : $|z_1 - z_2|$.

Section formula : $z = \dfrac{mz_2+nz_1}{m+n}$ (internal division), $z = \dfrac{mz_2-nz_1}{m-n}$ (external division)

1. amp(z) = θ is a ray emanating from the origin inclined at an angle θ to the x-axis.
2. $|z-a| = |z-b|$ is the perpendicular bisector of the line joining a to b.
3. The equation of a line joining z_1 & z_2 is given by, $z = z_1 + t(z_1 - z_2)$ where t is a real parameter.
4. The equation of circle having centre z_0 & radius ρ is :

   $|z-z_0| = \rho$ or $z\bar{z} - z_0\bar{z} - \bar{z}_0 z + \bar{z}_0\, z_0 - \rho^2 = 0$ which is of the form

   $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + k = 0$, k is real. Centre is $-\alpha$ & radius = $\sqrt{\alpha\bar{\alpha} - k}$

   Circle will be zero if $\alpha\bar{\alpha} - k \geq 0$....
5. if $|z_1 - z_1| + |z - z_2| = K > |z_1 - z_2|$ then locus of z is an ellipse whose focii are z_1 & z_2
6. if $\left|\dfrac{z-z_1}{z-z_2}\right| = k \neq 1, 0$ then locus of z is circle

   if $\big||z-z_1| - |z-z_2|\big| = K < |z_1 - z_2|$ then locus of z is a hyperbola, whose focii are z_1 & z_2

Quadratic Equation

1. Quadratic Equation : $ax^2 + bx + c = 0,\ a \neq 0$

   $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, The expression $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

   if α, β are the roots, then (a) $\alpha+\beta = -\dfrac{b}{a}$ (b) $\alpha\beta = \dfrac{c}{a}$

A quadratic equation whose roots are α & β, is $(x - \alpha)(x - \beta) = 0$ i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

2. Nature of Roots:

Consider the quadratic equation, $ax^2 + bx + c = 0$ having α, β as its roots; $D \equiv b^2 - 4ac$

- $D = 0$
  Roots are equal $\alpha = \beta = -b/2a$
- $D \neq 0$
  Roots are unequal
  - $a, b, c \in R$ & $D > 0$
    Roots are real
    - $a, b, c \in Q$ &
      D is a perfect square
      $\Rightarrow$ Roots are rational
      $\downarrow$
      $a = 1$, $b, c \in I$ & D is a perfect square
      $\Rightarrow$ Roots are integral.
    - $a, b, c \in Q$ &
      D is not a perfect square
      $\Rightarrow$ Roots are irrational
      i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$
    Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

3. Common Roots :

Consider two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

i. if two quadratic equations have both roots common, then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

ii. if only one root α is common, then $\alpha = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \dfrac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$

4. Range of Quadratic Expression $f(x) = ax^2 + bx + c$

Range in restricted domain : Given $x \in [x_1, x_2]$

a. if $-\dfrac{b}{2a} \notin [x_1, x_2]$ then, $f(x) \in \left[\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}\right]$

b. if $-\dfrac{b}{2a} \in [x_1, x_2]$ then, $f(x) \in \left[\min\left\{f(x_1) f(x_2), -\dfrac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\dfrac{D}{4a}\right\}\right]$

5. Location of Roots :

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a \cdot b \cdot c \in R$

i. Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) > x_0$.

ii. Conditions for both the roots of $f(x) = 0$ to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$; $f(x_0) > 0$ & $(-b/2a) < x_0$.

iii. Conditions that both roots of $f(x) = 0$ to lie on either side of the number 'x_0' (in other words the number 'x_0' lies between the roots of $f(x) = 0$) is $f(x_0) < 0$.
iv. Conditions that both roots of $f(x) = 0$ to be confined between the numbers x_1 and x_2, ($x_1 < x_2$) are $b_2 - 4ac \geq 0$; $f(x_1) > 0$; $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.
v. Conditions for exactly one root of $f(x) = 0$ to lie in the internal (x_1, x_2) i.e. $x_1 < x < x_2$ is $f(x_1) \cdot f(x_2) < 0$.

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Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

1. $A(\text{adj } A) = |A|\, I_n = (\text{adj } A)A$
2. $|\text{Adj } A| = |A|^{n-1}$ where A(adj A) is always a scalar matrix)
3. $\text{Adj}(\text{adj } A) = |A|^{n-2} A$
4. $|\text{adj}(\text{adj } A)|\ |A|^{(n-1)2}$
5. $\text{Adj}(A^T) = (\text{adj } A)^T$
6. $\text{Adj}(AB) = (\text{adj } B)\ (\text{adj } A)$
7. $\text{Adj } (A^m) = (\text{adj } A)^m,\ m \in N$
8. $\text{Adj}(kA) = k^{n-1}\ (\text{adj } A),\ k \in R$
9. $\text{Adj}(I_n) = I_n$

Properties of Inverse matrix :

Let A and B are two invertible matrices of the same order then

1. $(A^T)^{-1} = (A^{-1})^T$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^k)^{-1} = (A^{-1})^k,\ k \in N$
4. $\text{Adj}(A^{-1}) = (\text{adj } A)^{-1}$
5. $(A^{-1})^{-1} = A$
6. $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$
7. If $A = \text{diag}(a_1, a_2, \ldots..a_n)$ then $A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, a_3^{-1} \ldots\ldots a_n^{-1})$
8. A is symmetric matrix then A^{-1} is symmetric matrix.

Properties of Transpose :

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(AB)^T = B^TA^T$
4. $(kA)^T = k(A)^T$
5. $I^T = I$
6. $\text{Tr}(A) = \text{tr } (A)^T$
7. $(A_1A_2A_3\ldots..A_{n-1}A_n)^T = A_n^T A_{n-1}^T \ldots \ldots A_3^T A_2^T A_1^T$

Rank of matrix :

A number is said to be the rank of a m × n matrix A if

Every square sub matrix of order (r + 1) or more is singular and there exists at least one square sub matrix of order r which is non-singular.

The rank of matrix is the order of the highest order non-singular sub matrix.

1. The area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
   i. As area is a positive quantity, hence always take the absolute value of determinant.
   ii. If area of triangle is given, then use both positive and negative value of the determinant.
   iii. Three points A, B and C are collinear if area of triangle ABC is zero.

2. Cramer's Rule :

Consider, a system of linear equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \text{ and } x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

i. If $D \neq 0$ then the system has unique solution.

ii. If $D = 0$, the syetem may have infinite number of solutions or no solution.

iii. A system of lienar euqations is said to be consistent if it has at least are solution otherwise it is inconsistent.

Properties of Determinants :

1. The value of the determinant remains unchanged if its rows and columns are interchanged.
2. If any two rows (or columns) of a determinant are interchnaged, then sign of determinant chnages.
3. If any two rows (or columns) of a determinant are identical then the valie of determinant is zero.
4. If all the elements of any rwo (or column) of a matrix A are zero then $|A| = 0$
5. If each elements of a row (or a column) of a determinant is multiplied by a constant k then the value gets multiplied by k.
6. If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
7. If each element of any row or column of a determinant the equimultiples of correspoding elements of other rwo (or column) are added then value of determinant remains the same.

Important results on determinants :

1. $|AB| = |A||B|$ where A and B are square matrices of the same order.
2. $|A^n| = |A|^n$
3. If A, B and C are square matrices of the same order such that ith column (or row) of A is the sum of ith columns (or rows) of B and C and all other columns (or rows) of A, B and C are identical then $|A| = |B| + |C|$
4. $|I_n| = 1$ where I_n is identity matrix of order n.
5. $|O_n| = 0$ where O_n is a zero matrix of order n.
6. If $\Delta(x)$ be a 3rd order determinant having polynomials as its elements.
7. If $\Delta(a)$ has 2 rows (or columns) proportional then $(x - a)$ is a factor of $\Delta(x)$.
8. If $\Delta(a)$ has 3 rows (or columns) proportional then $(x - a)^2$ is a factor of $\Delta(x)$.
9. A square matrix A is non-singular, if $|A| \neq 0$ and singular if $|A| = 0$
10. Determinant of a skew-symmetric matrix of odd order is zero and of even order is a non-zero perfect square.
11. In general, $|B + C| \neq |B| + |C|$
12. Determinant of a diagonal matrix = Product of its diagonal elements.
13. Determinant of triangular matrix = Product of its diagonal elements.
14. If A is a non-singular matrix then $|A^{-1}| = \dfrac{1}{|A|} = |A|^{-1}$
15. Determinant of a orthogonal matrix is 1 or -1.
16. Determinant of a hermitian matrix is purely real.
17. If A and B are non-zero matrices and AB = 0 then it implies $|A| = 0$ and $|B| = 0$.

FORMULAE

1. **Arrangement** : Number of permutation of n different things taken r at a time =
$^{n}P_{r} = n(n-1)(n-2)....(n-r+1) = \frac{n!}{(n-r)!}$
2. **Circular Permutation :**
The number of circular permutations of n different things taken all at a time is; (n – 1)!
The number of ways of arranging n persons along a round table so that no person has the same two neghbours is $\frac{(n-1)!}{2}$.
3. Number of permutations of n distinct objects taken r at a time when a particular object is not taken in any arrangement is $^{n-1}P_r$
Number of permutations of r objects out of n distinct objects when a particular object is always included in any arrangement is $r \cdot {}^{n-1}P_{r-1}$
4. **Selection :** Number of combinations of n different things taken r at a time = $^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{^{n}P_{r}}{r!}$
5. The number of permutations of 'n' things, taken all at a time, when 'p' of them are similar & of one type, q of them are similar & of another type, 'r' of them are similar & of a third type & the remaining $n-(p+q+r)$are all different is $\frac{n!}{p!q!r!}$
6. **Selection of one or more objects**
a. Number of ways in which atleast one object be selected out of 'n' distinct objects is
$^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \ldots\ldots\ldots\ldots\ldots\ldots + {}^{n}C_n = 2^n - 1$
b. Number of ways in which atleast one object may be selected out of 'p' alike objects of one type 'q' alike objects of second type and 'r' alike of third type is
$(p + 1)\ (q + 1)\ (r + 1) - 1$
c. Number of ways in which atleast one object may be selected out of 'n' objects where 'p' alike of one type 'q' alike of second type and 'r' alike of third type and rest
$n - (p + q + r)$ are different, is
$(p + 1)\ (q+1)\ (r + 1)\ 2^{n-(p+q+r)} - 1$
7. **Multinomial Theorem :**
Coefficient of x^r in expansion of $(1 - x)^{-n} = {}^{n+r-1}C_r\ (n \in N)$

8. Let $N = p^a\, q^b\, r^c$ ……… where p, q, r ….. are distinct primes & a, b, c… are natural numbers then :
a. The total numbers of divisors of N including 1 & N is = $(a + 1)\ (b + 1)\ (c + 1)$ ……..
b. The sum of these divisors is =
$(p^0 + p^1 + p^2 + \ldots\ldots\ p^a)(q^0 + q^1 + q^2 + \ldots\ldots + q^b)\ (r^0 + r^1 + r^2 + \ldots\ldots + r^c....)$
c. Number of ways in which N can be resolved as a product of two factors is
$= \frac{1}{2}(a+1)(b+1)(c+1).....$ if N is a not perfect sqaure
$\frac{1}{2}\left[(a+1)(b+1)(c+1).....+1\right]$ if N is a perfect sqaure

d. Number of ways in which number N can be resolved into as a product of two factors which are relatively prime (or co-prime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

9. Dearrangement:

Number of ways in which 'n' letters can be put in 'n' corresponding envelopes such that no letter goes to correct envelope is n! $\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}......+(-1)^n\frac{1}{n!}\right)$

FORMULAE

The Principle of Mathematical Induction

The principle of mathematical induction is one such tool which can be used to prove a wide variety of mathematical statements. Such statement is assumed as P(n) associated with positive integer n, for which the correctness for the case n = 1 is examined. Then assuming the truth of P(k) for some positive integer k, the truth of
P(k+1) is established.

Steps:

Suppose there is a given statement P(n) involving the natural number n

i. Check if the statement is true for n = 1, i.e., P(1) is true,
   if true proceed to next step, else find the least value of n for which it holds true and then proceed to next step.
ii. Assume the statement is true for n = k (where k is some positive integer)
iii. Prove the statement is also true for n = k + 1, with the help of step (ii)
   If proved then, P(n) is true for all natural numbers n.

FORMULAE

1. **Statement of Binomial theorem** : if a, b ∈ R and n ∈ N, then

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots\ldots + {}^nC_r a^{n-r} b^r + \ldots + {}^nC_n a^0 b^n = \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r$$

2. **Properties of Binomial Theorem** :
   i. **General term** : $T_{r+1} = {}^nC_r a^{n-r} b^r$
   ii. **Middle term (s)** :
   a. If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.
   b. If n is odd, there are two middle terms, which $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

3. **Multinomial Theorem :** $(x_1 + x_2 + x_3 + \ldots\ldots x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} \ldots x_2^{r_2} \ldots x_k^{r_k}$

   Here total number of terms in the expansion = ${}^{n+k-1}C_{k-1}$

4. **Application of Binomial Theorem :**

   If $\left(\sqrt{A}+B\right)^n = I+F$ where I and n are positive integers n being odd and 0 < f < 1 then (I + f) f = k^n where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$.
   If n is an even integer, then (I + f) (1 – f) = k^n

5. **Properties of Binomial Coefficients :**
   i. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$
   ii. ${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots\ldots\ldots + (-1)^n\, {}^nC_n = 0$
   iii. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = 2^{n-1}$
   iv. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
   v. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

6. **Binomial Theorem For Negative Integer Or Fractional Indices**

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}x^r + \ldots, |x| < 1.$$

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}x^r$$

   Important series :
   $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots$
   $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$
   $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots.$
   $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots\ldots$

FORMULAE

An arithmetic progression (A.P.): a + d, a + 2d, ……….. a + (n – 1) d is an A.P.
Let a be the first term and d be the common differences of an A.P., then n^{th} term = $t_n = a + (n-1)d$
The sum of first n terms of are A.P.

$$S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+\ell]$$

r^{th} term of A.P. when sum of first r terms is given is $t_r = S_r - S_{r-1}$.
Properties of A.P.

i. If a, b, c are in A.P. ⇒ 2b = a + c & a, b, c are in A.P. ⇒ a + d = b + c
ii. Three numbers in A.P. can be taken as a – d a, a + d; four numbers in A.P. can be taken as a – 3d, a – d, a + d, a + 3d; five numbers in A.P. are a – 2d, a – d, a, a + d, a + 2d & six terms in A.P. are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.
iii. Sum of the terms of an A.P. equidistant from the beginning & end = sum of first & last term.

Arithmetic Mean (Mean or Average) (A.M.):
If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a,b,c are in A.P., is A.M. of a & c.

n - Arithmetic Means between Two Numbers:
if a, b are any two given numbers & a, A_1, A_2, ………, A_n, b are in A.P. then A_1, A_2,…… A_n are the n A.M.'s between a & b. $A_1 = a+\frac{b-a}{n+1}$, $A_2 = a+\frac{2(b-a)}{n+1}$, ……., $A_n = a+\frac{n(b-a)}{n+1}$

$\sum_{r=1}^{n} A_r = nA$ where A is the single A.M.between a & b.

Geometric Progression: a, ar, ar^2, ar^3, ar^4, ………. is G.P. with a as the first term & r as common ratio.

i. n^{th} term = $a\,r^{n-1}$
ii. Sum of the first n terms i.e. $S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & r \neq 1 \\ na & ,r=1 \end{cases}$
iii. Sum of an infinite G.P. when $|r|<1$ is given by $S_\infty = \frac{a}{1-r}\left(|r|<1\right)$.

Geometric Means (Means Proportional) (G.M.):
If a,b,c > 0 are in G.P., be is the G.M. between a & c, then $b^2 = ac$ n – Geometric Means between positive number a, b: if a, b are two given numbers & a, G_1, G_2, ……….,G_n,b are in G.P.. Then G_1, G_2, G_3,…, G_n are n G.M.s between a & b.
$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$,……., $G_n = a(b/a)^{n/n+1}$

Harmonic Mean (H.M.):
if a,b,c are in H.P., be is the H.M. between a & c, then b = $\frac{2ac}{a+c}$

H.M.H of $a_1, a_2, \ldots\ldots a_n$ is given by $\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right]$

Relation between means:
$G^2 = AH$, A.M. ≥ G.M. ≥ H.M and A.M = G.M. = H.M. if $a_1 = a_2 = a_3 = \ldots\ldots = a_n$

Important Results

i. $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$.

ii. $\sum_{r=1}^{n} k\, a_r = k\sum_{r=1}^{n} a_r$.

iii. $\sum_{r=1}^{n} k = nk$; where k is a constant.

iv. $\sum_{r=1}^{n} r = 1+2+3+....+n = \frac{n(n+1)}{2}$

v. $\sum_{r=1}^{n} r^2 = 1^2+2^2+3^2+......+n^2 = \frac{n(n+1)(2n+1)}{6}$

vi. $\sum_{r=1}^{n} r^3 = 1^3+2^3+3^3+........+n^3 = \frac{n^2(n+1)^2}{4}$

vii. $2\sum_{i<j=1}^{n} a_i a_j = (a_1+a_2+.....+a_n)^2 - (a_1^2+a_2^2+......+a_n^2)$

FORMULAE

Application of derivative

1. Equation of tangent and normal

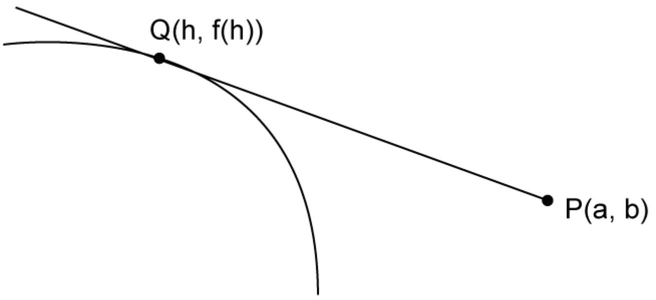
Tangent at (x_1, y_1) is given by $(y - y_1) = f'(x_1)(x - x_1)$; when $f'(x_1)$ is real.

And normal at (x_1, y_1) is $(y - y_1) = -\frac{1}{f'(x_1)}(x - x_1)$, when $f'(x_1)$ is non zero real.

2. Tangent from an external point

Given a point P (a, b) which does not lie on the curve y = f(x), then the equation of possible tangents to the curve y = f(x), passing through (a, b) can be found by solving for the point of contact Q.

$$f'(h) = \frac{f(h)-b}{h-a}$$



And equation of tangent is $y-b = \frac{f(h)-b}{h-a}(x-a)$

3. Length of tangent, normal, subtangent, subnormal

i. $PT = |k|\sqrt{1+\frac{1}{m^2}}$ = Length of Tangent

ii. $PN = |k|\sqrt{1+m^2}$ = Length of Normal

iii. $TM = \left|\frac{k}{m}\right|$ = Length of subtangent

iv. $MN = |km|$ = Length of subnormal

4. Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents (or normals) at the point of intersection of two curves (as shown in figure)

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

5. Shortest distance between two curves

Shortest distance between two non-intersecting differentials curves is always along their common normal (Whenever defined)

6. **Rolle's Theorem:**

If a function f defined on [a, b) is

i. Continuous on [a, b]
ii. derivable on (a, b) and
iii. f(a) = f (b)

Then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$

7. **Lagrange's Mean value Theorem (LMVT):**

If a function f defined on [a,b] is

i. Continuous on [a, b] and
ii. derivable on (a, b)

Then there exists at least one real numbers between a and b ($a < c < b$) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

8. **Maxima ad Minima**

i. A function has a local maxima at x = a if $f(x) \leq f(c)$, for every x in some open interval around x = c.
ii. A function has a local minima at x = c if $f(x) \geq f(c)$, for every x in some open interval around x = c.
iii. A function has a global maxima at x = a if $f(x) \leq f(c)$, for every x in the domain under consideration.
iv. A function has a global minima at x = c if $f(x) \geq f(c)$, for every x in the diomain under consideration.

9. **Useful Formulae of Mensuration to Remember:**

i. Volume of a cuboid = ℓbh.
ii. Surface area of cuboid = $2(\ell b + bh + h\ell)$
iii. Volume of cube = a^3
iv. Surface area of cube = $6a^2$
v. Volume of a cone = $\frac{1}{3}\pi r^2 h$
vi. Curved surface area of cone = $\pi r\ell$ (ℓ = Slant height)
vii. Curved surface area of a cylinder = $2\pi rh$.
viii. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$
ix. Volume of a sphere = $\frac{4}{3}\pi r^3$
x. Surface area of a sphere = $4\pi r^2$
xi. Area of a circular sector = $\frac{1}{2} r^2 \theta$ is in radians.
xii. Volume of a prism = (area of the base) × (height).
xiii. Lateral surface area of a prism = (perimeter of the base) × (height).
xiv. Total surface area of a prism = (lateral surface area) + 2 (area of the base)

(Note that lateral surfaces of a prism are all rectangle)

xv. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height)
xvi. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height)

(Note that slant surfaces of a pyramid are triangles).

Limit of function

1. Limit of function f(x) is said to exist as $x \rightarrow a$ when,

$\underset{h\to 0^+}{\text{Limit}}\ f(a-h) = \underset{h\to 0^+}{\text{Limit}}\ f(a+h) =$ some finite value M.

(Left hand limit) (Right hand limit)

2. Indeterminant Forms :

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty^{o}, 0^{o}$, and 1^{∞}.

3. Standard Limits :

$$\underset{x\to 0}{\text{Limit}}\ \frac{\sin x}{x} = 1, \underset{x\to 0}{\text{Limit}} \frac{\tan x}{x} = 1, \underset{x\to 0}{\text{Limit}}\ \frac{\sin^{-1} x}{x} = 1, \underset{x\to 0}{\text{Limit}} \frac{\tan^{-1} x}{x} = 1, \underset{x\to 0}{\text{Limit}}\ \frac{e^x - 1}{x} = 1, \underset{x\to 0}{\text{Limit}} \frac{\ell n(1+x)}{x} = 1$$

$$\underset{x\to 0}{\text{Limit}}\ (1+x)^{1/x} = e, \underset{x\to \infty}{\text{Limit}} \left(1+\frac{1}{x}\right)^x = e,\ \ \underset{x\to 0}{\text{Limit}} \frac{a^x - 1}{x} = \log_e a, a > 0,\ \ \underset{x\to 0}{\text{Limit}} \frac{x^n - a^n}{x-a} = na^{n-1}.$$

4. Limits Using Expansion

i. $a^x = 1 + \frac{x \ln a}{1!} + \frac{x^2 \ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + \ldots\ldots\ldots a>0$

ii. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$

iii. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$ for $-1<x \le 1$

iv. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$

v. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

vi. $\tan x = x + \frac{x^3}{3!} + \frac{2x^5}{15!} + \ldots\ldots\ldots$

vii. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

viii. $\sin^{-1} x = x + \frac{1^2}{3!} x^3 + \frac{1^2 \cdot 3^2}{5!} x^5 + \frac{1^2 \cdot 3^2 \cdot 5^2}{7!} x^7 + \ldots\ldots\ldots$

ix. for $|x| < 1,\ n \in R$ $(1 + x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2} x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} x^3 + \ldots\ldots\ldots\infty$

5. Limits of form 1∞, 0°, ∞°

Also for $(1)^{\infty}$ type of problems we can use following rules.

$\lim\limits_{x\to 0}(1+x)^{1/x} = e, \lim\limits_{x\to a}[f(x)]^{g(x)}$, where $f(x) \to 1$; $g(x) \to \infty$ as $x \to a$ = $\lim\limits_{x\to a} = \lim\limits_{e^x\to 0}[f(x)-1]g(x)$

6. Sandwich Theorem or Squeeze Play Theorem:

If f(x) ≤ g(x) ≤ h(x) ∀ × $\underset{x\to a}{\text{Limit}}\ f(x) = \ell = \underset{x\to a}{\text{Limit}}\ h(x)$ then $\underset{x\to a}{\text{Limit}}\ g(x) = \ell$.

Method of Differentiation

1. Differentiation of some elementary functions

i. $\frac{d}{dx}(x^n) = nx^{n-1}$

ii. $\frac{d}{dx}(a^x) = a^x \ell n\ a$

iii. $\frac{d}{dx}(\ell n|x|) = \frac{1}{x}$

iv. $\frac{d}{dx}(\log_a x) = \frac{1}{x\ell n\ a}$

v. $\frac{d}{dx}(\sin\ x) = \cos x$

vi. $\frac{d}{dx}(\cos x) = -\sin x$

vii. $\frac{d}{dx}(\sec\ x) = \sec x \tan x$

viii. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

ix. $\frac{d}{dx}(\tan x) = \sec^2 x$

x. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

2. Basic Theorems

i. $\frac{d}{dx}(f \pm g) = f'(x) \pm g'(x)$

ii. $\frac{d}{dx}(kf(x) = k\frac{d}{dx}f(x)$

iii. $\frac{d}{dx}(f(x).g(x)) = f(x)g'(x) + g(x)f'(x)$

iv. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

v. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

3. Derivative of Inverse Trigonometric Functions.

$\frac{d \sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{d\cos^{-1}x}{dx} = -\frac{1}{\sqrt{1-x^2}}$, for $-1<x<1$.

$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}, \frac{d\cot^{-1}x}{dx} = -\frac{1}{1+x^2}$ $(x \in R)$

$\frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \frac{d\,\text{cosec}^{-1}x}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$, for $x \in (-\infty,-1) \cup (1,\infty)$

4. **Differentiation using substitution**

Following substitutions are normally used to simplify these expression.

i. $\sqrt{x^2+a^2}$ by substituting x = a tan θ, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

ii. $\sqrt{a^2-x^2}$ by substituting x = a sin θ, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

iii. $\sqrt{x^2-a^2}$ by substituting x = a cos θ, where $\theta \in [0,\pi]$, $\theta\neq\frac{\pi}{2}$

iv. $\sqrt{\frac{x+a}{a-x}}$ by substituting x = a cos θ, where $\theta \in (0,\pi]$

5. **Logarithmic Differentiation**

i. a product of a number of functions
ii. a quotient of functions
iii. a function which is the power of other function i. e. $[f(x)]^{g(x)}$
then by taking logarithm we can simplify these functions and differentiate.
Such type of differentiation is known as logarithmic differentiation.

6. **Parametric Differentiation**

if y = f(θ) & x = g(θ)where θ is a parameter, then $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$

7. **Derivative of one function with respect to another**

Let y = f(x); z = g(x) then $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}$

8. If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f,g,h,l,m,n,u,v,w are differentiable functions of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

FORMULAE

Properties of definite integral

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$

4. $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x)+f(-x))\,dx = \begin{cases} 2\int_0^a f(x)dx & f(-x) = f(x) \\ 0 & f(-x) = -f(x) \end{cases}$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)dx$

6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

7. $\int_0^{2a} f(x)\,dx = \int_0^a (f(x)+f(2a-x))\,dx = \begin{cases} 2\int_0^a f(x)dx & f(2a-x) = f(x) \\ 0 & f(2a-x) = -f(x) \end{cases}$

8. if f(x) is a periodic function with period T, then

$\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z,\quad \int_a^{a+nT} f(x)\,dx = n\int_0^T f(x)\,dx,\ n \in z,\ a \in R$

$\int_{mT}^{nT} f(x)\,dx = (n\text{-}m)\int_0^T f(x)dx,\ m\ n\ \in z,\quad \int_{nT}^{a+nT} f(x)\,dx = \int_0^a f(x)dx, n \in z,\ a \in R$

$\int_{a+nT}^{b+nT} f(x)\ \ dx = \int_a^a f(x)\,dx,\ n \in z,\ a,b\ \in R$

9. if $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then $\int_a^b \Psi'(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

10. if $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

11. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

12. if $f(x) \ge 0$ on [a,b]then $\int_a^b f(x)\,dx \ge 0$

Leibnitz Theorem: if $F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$, then $\dfrac{dF(x)}{dx} = h'(x)\,f(h(x)\text{-}g'(x)f(g(x))$

Indefiinite Intergration

1. If f & a are functions of x such that g'(x) = f(x) then,

$\int f(x)dx = g(x) + c \Leftrightarrow \dfrac{d}{dx}\{g(x)+c\} = f(x),$ where c is called the **constant of intergration**.

2. Standard Formula :

i. $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$

ii. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a}\ell n(ax+b) + c$

iii. $\int e^{ax+b} dx = \dfrac{1}{a} e^{ax+b} + c$

iv. $\int a^{px+q} dx = \dfrac{1}{p}\dfrac{a^{px+q}}{\ell na} + c; a > 0$

v. $\int \sin\ (ax+b)\ dx = -\dfrac{1}{a}\ \cos\ (ax+b)+c$

vi. $\int \cos(ax+b)dx = \dfrac{1}{a}\ \sin\ (ax+b)+c$

vii. $\int \tan\ (ax+b)dx = \dfrac{1}{a}\ell n\ \sec(ax+b)+c$

viii. $\int \cot(ax+b)dx = \dfrac{1}{a}\ell n\ \sin\ (ax+b)+c$

ix. $\int \sec^2(ax+b)dx = \dfrac{1}{a}\tan\ (ax+b)+c$

x. $\int \operatorname{cosec}^2(ax+b)dx = -\dfrac{1}{a}\cot\ (ax+b)+c$

xi. $\int \sec x\ dx = \ell n\ (\sec x + \tan x) + c$ **OR** $\ell n \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)+c$

xii. $\int \operatorname{cosec} x\ dx = \ell n\ (\operatorname{cosec}x - \cot x)+c$ or $\ell n \tan\dfrac{x}{2}+c$ OR $-\ell n\ (\operatorname{cosec}x+\cot x)+c$

xiii. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}+c$

xiv. $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+c$

xv. $\int \dfrac{dx}{|x|\sqrt{x^2-a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a}+c$

xvi. $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \ell n\left[x+\sqrt{x^2+a^2}\right]+c$

xvii. $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \ell n\left[x+\sqrt{x^2-a^2}\right]+c$

xviii. $\int \dfrac{dx}{a^2-x^2} = \dfrac{1}{2a}\ell n\left|\dfrac{a+x}{a-x}\right|+c$

xix. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ell n\left|\dfrac{x-a}{x+a}\right|+c$

xx. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

xxi. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ell n\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) + c$

xxii. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ell n\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c$

3. **Intergration by Substitutions**

If we substitute f(x) = t then f'(x) dx = dt

4. **Integration by Part:**

$$\int (f(x)g(x))dx = f(x)\int (g(x))dx - \int\left(\frac{d}{dx}(f(x))\int (g(x))dx\right)dx$$

5. **Integration of type** $\int \frac{dx}{ax^2 + bx + c}, \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \sqrt{ax^2 + bx + c}\ dx$

Make the substitution $x + \frac{b}{2a} = t$

6. **Integration of type**

$\int \frac{px + q}{ax^2 + bx + c}\, dx,\ \int \frac{px + q}{\sqrt{ax^2 + bx + c}}\, dx,\ \int (px + q)\sqrt{ax^2 + bx + c}\ dx$

Make the substitution $x + \frac{b}{2a} = t$, then split the integral as some of two integrals one containing the linear term and the other containing constant term.

7. **Integration of trigonometric functions**

i. $\int \frac{dx}{a + b\sin^2 x}$ OR $\int \frac{dx}{a + b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$ put tan x = t.

ii. $\int \frac{dx}{a + b\sin x}$ OR $\int \frac{dx}{a + b\cos x}$ OR $\int \frac{dx}{a + b\sin x + c\cos x}$ put $\tan\frac{x}{2} = t$

iii. $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}$ dx.Express Nr $\equiv$ A(Dr) + B$\frac{d}{dx}$ (Dr)+c & proceed.

8. **Integration of type**

$\int \frac{x^2 \pm 1}{x^4 + kx^2 + 1}$ dx where k is any constant.

Divide Nr & Dr by x^2 & put $x \mp \frac{1}{x} = t$.

9. **Integration of type**

$\int \frac{dx}{(ax + b)\sqrt{px + q}}$ OR $\int \frac{dx}{(ax^2 + bx + c)\sqrt{px + q}}$; put $px+q=t^2$

10. **Integration of type**

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put $x=\frac{1}{t}$

FORMULAE

1. Order and degree of differential equation

$$f(x,y)\left[\frac{d^my}{dx^m}\right]^p + g(x,y)\left[\frac{d^{m-1}y}{dx^{m-1}}\right]^q + h(x,y)\left[\frac{d^{m-2}y}{dx^{m-2}}\right]^t + \ldots\ldots$$

Order: highest derivative $\rightarrow$ m

Degree: exponent of the highest derivative $\rightarrow$ n

2. Linear differential equation of first order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{I.F.} = e^{\int P(x)dx}$$

Solution :

$$y(\text{I.F.}) = \int Q(x)(\text{I.F.})dx + C$$

3. Some exact differentials

i. $xdy + ydx = d(xy)$

ii. $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

iii. $\frac{xdy - ydx}{y^2} = d\left(-\frac{x}{y}\right)$

iv. $\frac{xdx + ydy}{xy} = \frac{d(xy)}{xy} = d(\ln\ xy)$

v. $\frac{xdy - ydx}{xy} = d\left(\ln\left(\frac{y}{x}\right)\right)$

vi. $\frac{ydx - xdy}{xy} = d\left(\ln\left(\frac{x}{y}\right)\right)$

vii. $\frac{xdy + ydx}{x^2y^2} = d\left(-\frac{1}{xy}\right)$

viii. $\frac{2(xdx + ydy)}{x^2 + y^2} = \left(\ln\left(x^2 + y^2\right)\right)$

ix. $\frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

x. $d\left(\frac{e^y}{x}\right) = \left(\frac{xe^y dy - e^y dx}{x^2}\right)$

xi. $d\left(\frac{e^x}{y}\right) = \left(\frac{ye^x dy - e^x dy}{y^2}\right)$

xii. $\frac{dx + dy}{x + y} = d(\ln(x + y))$

FORMULAE

Circle

1. Intercepts made by Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ **on the Axes** :
   a. $2\sqrt{g^2 - c}$ on x - axis
   b. $2\sqrt{f^2 - c}$ on y-aixs

2. Parametric Equations of a Circle : $x = h + r\cos\theta;\ y = k + r\sin\theta$

3. Tangent :
   a. Slope form : $y = mx \pm a\sqrt{1+m^2}$
   b. Point form : $xx_1 + yy_1 = a^2$ or T = O
   c. Parametric form : $x\cos\alpha + y\sin\alpha = a$.

4. Pair of Tangents from a Point : $SS_1 = T^2$.

5. Length of a Tangent : Length of tangent is $\sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ at $P(x_1, y_1)$

6. The equation of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with $P(x_1, y_1)$ as the mid point of the chord is given by
   $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) = x_1^2 + y_1^2 + 2gx_1 + 2fy_1$
   $T = S_1$

7. Director Circle : $x^2 + y^2 = 2a^2$ for $x^2 + y^2 = a^2$

8. Chord of Contact : T = 0
   i. Length of chord of contact = $\dfrac{2LR}{\sqrt{R^2 + L^2}}$
   ii. Area of the triangle formed by the pair of the tangents & its chord of contact = $\dfrac{RL^3}{R^2 + L^2}$
   iii. Tangent of the angle between the pair of tangents from (x_1, y_1) = $\left(\dfrac{2RL}{L^2 - R^2}\right)$
   iv. Equation of the circle circumscribing the triangle PT_1T_2 is: $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

9. Condition of orthogonality of Two Circles: $2g_1g_2 + 2f_1f_2 = c_1 + c_2$.

10. Radical Axis : $S_1 - S_2 = 0$ i.e., $2(g_1 - g_2)x + 2(f_1 - f_2)y + (c_1 - c_2) = 0$

11. Family of Circles: $S_1 + KS_2 = 0$, $S + KL = 0$.

Parabola

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4by$	$x^2 = -4by$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	(a, 0)	(-a, 0)	(0, a)	(0, -a)
Equation of the directrix	x = -a	x = a	y = -a	y = a
Equation of the axis	y = 0	y = 0	x = 0	x = 0
End points of the Latus rectum	(a, ±2a)	(-a, ±2a)	(±2b, b)	(±2b, -b)
Length of the Latus rectum	4a	4a	4b	4b
Focal distance of a point P(x, y)	a + x	a – x	b + y	b – y
Equation of tangent in slope form	$y = mx + \frac{a}{m}$	$y = mx - \frac{a}{m}$	$y = mx - am^2$	$y = mx + am^2$
Parametric Coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Equation of tangent in parametric form	$y = \frac{x}{t} + at$	$y = \frac{x}{t} - at$	$y = \frac{x}{t} - b\frac{1}{t^2}$	$y = \frac{x}{t} + b\frac{1}{t^2}$
Equation of normal in parametric form	$tx + y = 2at + at^3$	$tx + y = 2at - at^3$	$tx + y = 3bt^2$	$tx + y = bt^2$

If the vertex of the parabola is at the point B(h, k) and its latus rectum is of length 4a, then its equation is

$(y - k)^2 = 4a(x - h)$ or $(y - k)^2 = -4a(x - h)$

$(x - h)^2 = 4a(y - k)$ or $(x - h)^2 = -4a(y - k)$

Ellipse

1.

	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a < b$
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the vertices	(a, 0) and (-a, 0)	(0, b) and (0, -b)
Coordinates of foci	(ae, 0) and (-ae, 0)	(0, be) and (0, -be)
Length of major axis	2a	2b
Length of minor axis	2b	2a
Equation of major axis	y = 0	x = 0
Equation of minor axis	x = 0	y = 0
Equation of the directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$x = \frac{b}{e}$ and $x = -\frac{b}{e}$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Length of the latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	a ± ex	b ± ey

If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

2. **Auxiliary Circle** : $x^2 + y^2 = a^2$

3. **Parametric Representation** : $x = a \cos \theta$ & $y = b \sin \theta$

4. **Position of a Point w.r.t. an Ellipse** :

   The point P (x_1, y_1) lies outside, inside or on the ellipse according as; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 >< \text{or} = 0$.

5. **Line and an Ellipse**: The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in two points real, coincident or imaginary according as c^2 is < = or > $a^2m^2 + b^2$.

6. **Tangents** : Slope form: $y = mx \pm \sqrt{a^2m^2+b^2}$, Point form : $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$,

   Parametric form : $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

7. **Normal:** $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$, $ax.\sec\theta$ - by $\operatorname{cosec}\theta = (a^2-b^2)$, $y = mx-\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$

8. **Director Circle** : $x^2 + y^2 = a^2 + b^2$

Hyperbola

1. **Standard Equation:** Standard equation of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $b^2 = a^2$

	Horizonatal hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	**Vertical hyperbola** $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the centre	(a, 0) and (-a, 0)	(0, a and (0, -a)
Coordinates of foci	(±ae, 0)	(0, ±ae)
Length of the transverse axis	2a	2a
Length of conjugate axis	2b	2b
Equations of the directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{a}{e}$
Eccentricity	$b^2 = a^2(e^2 - 1)$	$b^2 = a^2(e^2 - 1)$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$

Equation of the transverse axis	y = 0	x = 0
Equation of conjugate axis	x = 0	y = 0
Focal distances of any point (x_1, y_1)	ex ± a	ey ± a

2. **Auxiliary Circle :** $x^2 + y^2 = a^2$

3. **Parametric Representation :** $x = a \sec \theta$ & $y = b \tan \theta$

4. **Position of A point 'P' w.r.t. A Hyperbola :**

$S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 >, = \text{or} < 0$ according as the point (x_1, y_1) lies inside, on or outside the curve.

5. **Tangents :**
   i. **Slope form** : $y = m \pm \sqrt{a^2m^2 - b^2}$
   ii. **Point form** : at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$
   iii. **Parametric form** : $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$

6. **Normals :-**
   a. at the point (x_1, y_1) is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$.
   b. at the point P (a sec θ, b tan θ) is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$
   c. Equation of normals in terms of its slope 'm' are $y = mx \pm \frac{(a^2 + b^2)m}{\sqrt{a^2 - b^2m^2}}$

7. **Asymptotes** : $\frac{x}{a} + \frac{y}{b} = 0$ and $\frac{x}{a} - \frac{y}{b} = 0$. Pair of asymptotes : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

8. **Rectangular Or Equilateral Hyperbola**: $xy = c^2$, eccentricity is $\sqrt{2}$ **.**
   Vertices: (±C, ±C); Focii: $(\pm\sqrt{2}c, \pm\sqrt{2}c)$. Directrices: $x + y = \pm\sqrt{2}\ c$
   Latus Rectum (l): $\ell = 2\sqrt{2}\ c$ = T.A. = C.A.
   Parametric equation $x = ct$, $y = c/t$, $t \in R - \{0\}$
   Equation of the tangent at P $(x_1'y_1)$ is $\frac{x}{x_1} + \frac{y}{y_1} = 2$ & at P(t) is $\frac{x}{t} + t\,y = 2\,c$
   Equation of the normal at P (t) is $x\,t^3 - y\,t = c\,(t^4 - 1)$
   Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

FORMULAE

3-Dimension

1. **Vector representation of a point :** Position vector of point : P(x,y,z) is $x\hat{i}+y\hat{j}+zk$.

2. **Distance formula :** $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$, AB=$|\overrightarrow{OB}-\overrightarrow{OA}|$

3. **Distance of P from coordinate axes :** $PA=\sqrt{y^2+z^2}$, PB=$\sqrt{z^2+x^2}$, $PC=\sqrt{x^2+y^2}$

4. **Section Formula :** $x=\dfrac{mx_2+nx_1}{m+n}, y=\dfrac{my_2+ny_1}{m+n}, z=\dfrac{mz_2+nz_1}{m+n}$

   Mid-point: $x=\dfrac{x_1+x_2}{2}, y=\dfrac{y_1+y_2}{2}, z=\dfrac{z_1+z_2}{2}$

5. **Direction Cosines And Direction Ratios**
   i. Direction Cosines: Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then cos α, cosβ, cos γ are called the direction cosines of the line. The direction cosines are usually denoted by (ℓ, m, n). Thus ℓ = cos α, m = cos β, n = cos γ.
   ii. If ℓ, m, n be the direction cosines of a line, then $\ell^2+m^2+n^2=1$
   iii. Direction ratios: Let a, b, c be the proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.
   iv. If ℓ, m, n be the direction cosines and a,b,c be the direction ratios of a vector, then
   $$\ell=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$
   v. If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2) then the direction ratios of line PQ are, a = $x_2 - x_1$, b = $y_2 - y_1$ & c = $z_2 - z_1$ and the direction cosines of line PQ are $\ell=\dfrac{x_2-x_1}{|PQ|}$, m = $\dfrac{y_2-y_1}{|PQ|}$ and n = $\dfrac{z_2-z_1}{|PQ|}$

6. **Angle Between Two Line Segments :**
   $$\cos\theta = \left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$
   The line will be perpendicular of $a_1a_2 + b_1b_2 + c_1c_2 = 0$, parallel if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

7. **Projection of a line segment on a line**
   if $P(x_1, y_1, z_1)$ and Q (x_2, y_2, z_2) then the projection of PQ on a line having direction cosines ℓ, m, n is $|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|$

8. Equation of A Plane: General form ax + by + cz + d = 0, where a, b, c are not all zero, a, b, c, d ∈ R.

i. Normal form : $\ell x + my + nz = p$

ii. Plane through the point (x_1,y_1,z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

iii. Intercept Form : $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

iv. Vector form : $\left(\vec{r}-\vec{a}\right)\cdot\vec{n}=0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

v. Any plane parallel to the given plane ax + by + cz + d = 0 is ax + by + cz + λ = 0

Distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $= \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

(i) Equation of a plane passing through a given point & parallel to the given vectors :

$\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ (parametric form) where λ & μ are scalars. or $\vec{r}\cdot\left(\vec{b}\times\vec{c}\right)=\vec{a}\left(\vec{b}\times\vec{c}\right)$

(non parametric form)

9. A Plane & A Point

i. Distance of the point (x', y', z') from the plane ax + by + cz + d = 0 is given by $\frac{ax'+by'+cz+d}{\sqrt{a^2+b^2+c^2}}$

ii. Length of the perpendicular from a point $\left(\vec{a}\right)$ to plane $\vec{r}\cdot\vec{n}=d$ is given by p = $\frac{\left|\vec{a}\cdot\vec{n}-d\right|}{\left|\vec{n}\right|}$

iii. Foot (x', y', z') of perpendicular drawn from the point (x_1, y_1, z_1) to the plane ax + by + cz + d = 0 is given by $\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

iv. **To Find image of a point w.r.t. a plane :**

Let P (x_1, y_1, z_1) is a given point and ax + by + cz + d = 0 is given plane Let (x', y', z') is the image point. Then $\frac{x'-x_1}{a}=\frac{y'-y_1}{b}=\frac{z'-z_1}{c}=-2\frac{(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$

10. Angle Between Two Planes :

$\cos\theta = \left|\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}\right|$

Planes are perpendicular if aa' + bb' + cc' = 0 and planes are parallel if $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$

The angle θ between the planes $\vec{r}\cdot\vec{n}=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is given by, $\cos\theta=\frac{\vec{n}_1\cdot\vec{n}_2}{\left|\vec{n}_1\right|\cdot\left|\vec{n}_2\right|}$

Planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2=0$ & Planes are parallel if $\vec{n}_1=\lambda\vec{n}_2\cdot\lambda$ is a scalar

11. Angle Bisectors

i. The equations of the planes bisecting the angle between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

ii. Bisector of acute/obtuse angle: First make both the constant terms positive. Then

$a_1a_2+b_1b_2+c_1c_2 > 0 \Rightarrow$ origin lies on obtuse angle

$a_1a_2+b_1b_2+c_1c_2 < 0 \Rightarrow$ origin lies in acute angle

12. Family of Planes

i. Any plane through intersection of $a_1x+b_1y+c_1z+d_1=0$ & $a_2x+b_2y+c_2z+d_2=0$ is $a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$

ii. The equation of plane passing through the intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ & $\vec{r}\cdot\vec{n}_2 = d_2$ is $\vec{r}\cdot(n_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

13. Area of triangle: From two vector $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

14. Volume of a Tetrahedron : Volume of a tetrahedron with vertices A (x_1, y_1, z_1) B(x_2, y_2, z_2), C(x_3, y_3, z_3) and D (x_4, y_4, z_4) is given by $V = \frac{1}{6}\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]$

A Line

1. Equation of a Line

i. A straight line is intersection of two planes.

   a. It is represented by two planes $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2=0$

   b. Family of planes : $a_1x+b_1y+c_1z+d_1 + k(a_2x+b_2y+c_2z+d_2) = 0$

ii. Symmertric form : $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$

iii. Vector equation : $\vec{r} = \vec{a} + \lambda\vec{b}$

iv. Reduction of Cartesian form of equation of a line to vector form & vice versa.

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Leftrightarrow \vec{r} = (x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$$

2. Angle Between A plane And A Line :

i. if θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$,

then $\sin\theta = \left|\frac{a\ell+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{\ell^2+m^2+n^2}}\right|$

ii. Vector form : if θ is the angle between a line $\vec{r} = (\vec{a}+\lambda\vec{b})$ and $\vec{r}\cdot\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right]$

iii. Condition for perpendicularity $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$, $\vec{b}\times\vec{n} = 0$

iv. Condition for parallel $a\ell + bm + cn = 0$ $\quad \vec{b}\cdot\vec{n} = 0$

3. **Condition For A Line to Lie In A Plane**

i. Cartesian form: Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

ii. Vector form : Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}\cdot\vec{n} = d$ if $\vec{b}\cdot\vec{n} = 0$ & $\vec{a}\cdot\vec{n} = d$

4. **Skew Lines:**

i. The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines. if $\Delta \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} \neq 0$, then lines are skew

ii. Vector Form : For lines $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda\overrightarrow{b_2}$ to be skew $\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right)\cdot\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right) \neq 0$

iii. Shortest distance between lines $\vec{r} = \overrightarrow{a_1} + \lambda\vec{b}$ & $\vec{r} = \overrightarrow{a_2} + \mu\vec{b}$ is $d = \left|\frac{\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right)\times\vec{b}}{|\vec{b}|}\right|$

5. **Sphere**

General equation of a sphere is $x_2 + y_2 + z_2 + 2ux + 2vy + 2wz + d = 0$. $(-u, -v, -w)$ is the centre and $\sqrt{u^2 + v^2 + w^2 - d}$ is the radius of the sphere.

Sphere in Diameter form :

The equation of the sphere on the line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

As diameter is given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Straight Line

1. **Distance formula** : $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

2. **Section Formula** : $x = \frac{mx_2 \pm nx_1}{m \pm n}$; $y = \frac{my_2 \pm ny_1}{m \pm n}$

3. **Centroid, Incentre & Excentre** :

Centroid G $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, Incentre I $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

Excentre $I_1 \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c} \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

4. **Area of a Triangle :**

$$\Delta\, ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

5. **Slope Formula :**

i. Line joining two points $(x_1 y_1)\&(x_2 y_2)$, $m=\dfrac{y_1-y_2}{x_1-x_2}$

6. Condition of collinearity of three points : $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

7. Angle between two straight lines : $\tan\ \theta = \left|\dfrac{m_1-m_2}{1+m_1 m_2}\right|$

8. Two Lines : $ax + by + c = 0$ and $a'x + b'y + c' = 0$ two lines

1. Parallel if $\dfrac{a}{a'} = \dfrac{b}{b'} \neq \dfrac{c}{c'}$
2. Distance between two parallel lines = $\left|\dfrac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$
3. Perpendicular : if $aa' + bb' = 0$

9. A point and line :

1. Distance between point and line **=** $\left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$
2. Reflection of a point about a line **:** $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = -2\dfrac{ax_1+by_1+c}{a^2+b^2}$
3. Foot of the perpendicular from a point on the line is $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = -\dfrac{ax_1+by_1+c}{a^2+b^2}$

10. Bisectors of the angles between two lines : $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

11. Condition of Concurrency : of three straight lines $a_i x + b_i y + c_i = 0$, I = 1,2,3 is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

12. A Pair of straight lines through origin : $ax^2 + 2hxy + by^2 = 0$

If θ if the acute angle between the pair of straight lines, then the $\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$

FORMULAE

1. Position Vector of a Point:

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B - pv of A.

DISTANCE FORMULA: Distance between the two points A $(\vec{a})$ and B$(\vec{b})$ is AB = $|\vec{a} - \vec{b}|$

SECTION FORMULA : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}$. Mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$.

2. Scalar Product of Two Vectors $\vec{a}, \vec{b} = |\vec{a}||\vec{b}|$ cos θ, where $|\vec{a}|, |\vec{b}|$ are magnitude of $\vec{a}$ and $\vec{b}$ Respectively and θ is the angle between $\vec{a}$ and $\vec{b}$

1. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = k\cdot k = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot k = k\cdot\hat{i} = 0$ projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

2. if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3k$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3k$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

   The angle ϕ between $\vec{a}$ & $\vec{b}$ is given by cos ϕ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, $0 \le \phi \le \pi$ $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \ne 0\ \vec{b} \ne 0)$

3. Vector Product of Two Vectors:

1. if $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b}$ = $|\vec{a}||\vec{b}|\sin\theta\vec{n}$, where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}, \vec{b}$ & $\vec{n}$ forms a right handed screw system.
2. Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram, whose adjacent sides are represented by $\vec{a}$ & $\vec{b}$
3. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = k\times k = \hat{0}; \hat{i}\times\hat{j} = k, \hat{j}\times k = \hat{i}, k\times\hat{i} = \hat{j}$
4. if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3k$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3k$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
5. $\vec{a}\times\vec{b} = \vec{O} \Leftrightarrow$ $\vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \ne 0, \vec{b} \ne 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar
6. Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $n = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

i. if $\vec{a}, \vec{b}$ & $\vec{c}$ are the pv's of 3 points A,B & C then the vector area of a triangle ABC= $\dfrac{1}{2}\left[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}\right]$. The points A,B & C are collinear if $\bar{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a} = \vec{0}$

ii. Area of any quadrilateral whose diagonal vectors are $\overrightarrow{d_1}$ & $\overrightarrow{d_2}$ is given by $\dfrac{1}{2}\left|\overrightarrow{d_1}\times\overrightarrow{d_2}\right|$

iii. Lagrange Identity : $\left(\vec{a}\times\vec{b}\right)^2 = |\vec{a}|^2 |\vec{b}|^2 - \left(\vec{a}\vec{b}\right)^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

4. **Scalar Triple Product :**

i. The Scalar triple product of three vectors $\bar{a}, \bar{b} \& \bar{c}$ is defined as : $\vec{a}\times\left(\vec{b}\cdot\vec{c}\right) = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$

ii. Volume of tetrahedron $V = \left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$

iii. In a scalar triple product the position of dot & cross can be interchanged i.e.
$\vec{a}.\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\times\vec{b}\right).\vec{c}$ OR $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right]$

iv. $\vec{a}.\left(\vec{b}\times\vec{c}\right) = -\vec{a}\left(\vec{c}\times\vec{b}\right)$ i.e. $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\left[\vec{a}\ \vec{c}\ \vec{b}\right]$

v. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3 k; \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3 k \& \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3 k$ then $\left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

in general, if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}; \vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n} \& \vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{l}\ \vec{m}\ \vec{n}\right]$; where $\vec{\ell}, \vec{m} \& \vec{n}$ are non coplanar vectors

vi. if $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow$ $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

vii. Volume of tetrahedron OABC with O as origin & $A\left(\vec{a}\right), B\left(\vec{b}\right)$ and $C\left(\vec{c}\right)$ be the vertices

$= \left|\frac{1}{6}\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]\right|$

viii. The position vector of the centroid of a tetrahedron if the pv's of its vertices are $\vec{a}, \vec{b}, \vec{c} \& \vec{d}$ are given by $\frac{1}{4}\left[\vec{a} + \vec{b} + \vec{c} + \vec{d}\right]$

5. **Vector Triple Product :** $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{a}\cdot\vec{b}\right)\vec{c}, \left(\vec{a}\times\vec{b}\right)\times\vec{c} = \left(\vec{a}\cdot\vec{c}\right)\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\vec{a}$

i. $\left(\vec{a}\times\vec{b}\right)\times\vec{c} \neq \vec{a}\times\left(\vec{b}\times\vec{c}\right)$, in general

6. **Reciprocal System of Vectors**

if $\vec{a}, \vec{b}, \vec{c} \& \vec{a}', \vec{b}', \vec{c}'$ two sets of non coplanar vectors such that $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' =$ then the two systems are called Reciprocal System of vectors, where $\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{b}' = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}, \vec{c} = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$

FORMULAE

1. **Classical (A priori) Definition of Probability :**
If an experiment results in a total of (m + n) outcomes which are equally likely and mutually exclusive with one another and if 'm' outcomes are favorable to an event 'A' while 'n' are unfavorable, then the probability of occurrence of the event 'A' = $P(A)=\frac{m}{m+n}=\frac{n(A)}{n(S)}$.
We say that odds in favour of 'A' are m: n, while odds against 'A' are n: m.
$P\left(\bar{A}\right)=\frac{n}{m+n}=1-P(A)$

2. **Addition theorem of probability** : P (A ∪ B) P(A) + P(B) – P (A ∩ B)
De Morgan's Laws: (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$
Distribute Laws: (a) A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) (b) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)
i. P (A or B or C) = P (A) + P(B) + P(C) – P (A ∩ B) – P (B ∩ C) – P (C ∩ A) + P (A ∩ B ∩ C)
ii. P (at least two of A,B,C occur) = P (B ∩ C) + P (C ∩ A) + P (A ∩ B) – 2P (A ∩ B ∩ C)
iii. P (exactly two of A,B,C occur) = P (B ∩ C) + P (C ∩ A) + P (A ∩ B) – 3P (A ∩ B ∩ C)
iv. P (exactly one of A,B,C occur) =
v. P (A) + P (B) + P(C) – 2P (B ∩ C) – 2P (C ∩ A) – 2P (A ∩ B) + 3P (A ∩ B ∩ C)

3. **Conditional Probability** : P(A/B) = $\frac{P(A\cap B)}{P(B)}$

4. **Binomial Probability Theorem**
If an experiment is such that the probability of success or failure does not change with trials, then the probability of getting exactly r success in n trials of experiment is ${}^nC_r\, p^r\, q^{n-r}$ where 'p' is the probability of a success and q is the probability of a failure. Note that p + q = 1.

5. **Expectation :**
If a value M_i is associated with a probability of p_i then the expectation is given by $\sum p_i M_i$

6. **Total Probability Theorem :** $P(A)=\sum_{i=1}^{n} P(B_i).P(A/B_i)$

7. **Bayes' Theorem :**
If an event A can occur with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots\ldots B_n$ and the probabilities $P(A/B_1), P(A/B_2)$ ..... $P(A/B_n)$ are known, then $P(B_i/A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$

$B_1, B_2, B_3 \ldots\ldots\ldots\ldots\ldots B_n$
$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup \ldots\ldots\ldots\ldots \cup (A \cap B_n)$
$P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots\ldots\ldots\ldots + P(A \cap B_n) = \sum_{i=1}^{n} P(A\cap B_i)$

8. Binomial Probability Distribution :

i. Mean of any probability distribution of a random variable is given by : $\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i$

ii. Variance of a random variable is given by, $\sigma^2 = \sum (x_i - \mu)^2 . P_i = \sum p_i x_i^2 - \mu^2$

FORMULAE

Basics

Intervals:

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, b ∈ R such that a < b. we can define four types of intervals as follows:

		Symbols Used
i.	Open interval: (a, b) = {x:a < x < b} i.e. end points are not included	( ) or] [
ii.	Closed interval: [a, b] = {x : a ≤ x ≤ b} i.e. end points are also included. This is possible only when both a and b are finite.	[ ]
iii.	Open-closed interval : (a, b] = {x : a < x ≤ b}	( ] or]]
iv.	Closed – open interval : [a, b) = x : a ≤ x < b}	[ ) or [[

1. Properties of Modulus:

For any a, b ∈ R

$$|a| \geq 0, \quad |a| = |-a|, \quad |a| \geq a, |a| \geq -a, \quad |ab| = |a|\ |b|, \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|},$$

$$|a+b| \leq |a| + |b|, \quad |a-b| \geq \big||a| - |b|\big|$$

2. Trigonometric Identities :

i. $\text{Sin}^2 x + \cos^2 x = 1$

ii. $1 + \tan^2 x = \sec^2 x$

iii. $1 + \cot^2 x = \text{cosec}^2 x$

3. Trigonometric Functions of Sum or Difference of Two Angles:

a. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\quad \therefore 2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ and $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

b. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\therefore 2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ and $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

c. $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A + B) \cdot \sin(A - B)$

d. $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)$

e. $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

f. $\tan(A + B + C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

4. Factorisation of the Sum or Difference of Two Sines or Cosines:

a. $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

b. $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

c. $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

d. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

5. Multiple and Sub-multiple Angles:

a. $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$; $2\cos^2\frac{\theta}{2} = 1 + \cos\theta, 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$

b. $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$, $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$

c. $\sin 3A = 3\sin A - 4\sin^3 A$

d. $\cos 3A = 4\cos^3 A - 3\cos A$

e. $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

6. Important Trigonometric Ratios:

a. $\sin n\pi = 0$; $\cos n\pi = (-1)$; $\tan\pi = 0$, where $n \in I$

b. $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$;

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$; $\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^\circ$

c. $\sin\frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ & $\cos 36^\circ$ $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

7. Range of Trigonometric Expression:

$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

8. Sine and Cosine Series:

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos\left(\alpha + \overline{n-1}\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Trigonometric Equations

9. Principal Solutions: Solutions which lie in the interval $[0, 2\pi)$ are called principal solutions.
General Solution:

i. $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], n \in I.$

ii. $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi], n \in I.$

iii. $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), n \in I.$

iv. $\sin^2\theta = \sin^2\alpha, \cos^2\theta = \cos^2\alpha, \tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha.$

10. Logarithmic formulae

i. If $a^x = b$ then $\log_a b = x$
ii. $\text{Log}_a 1 = 0$ where $a > 0, a \neq 1$
iii. $\text{Log}_a a = 1$ where $a > 0, a \neq 1$
iv. $a^{\log_a x} = x$, where $a > 0, a \neq 1$
v. $\log_a mn = \log_a m + \log_a n$
vi. $\log_a (m/n) = \log_a m - \log_a n$
vii. $\log_a m^n = n\log_a m$, where $m, a > 0, a \neq 1$
viii. Change of base : $\log_a b = \dfrac{\log_m b}{\log_m a}$ where $a, b, m > 0, a \neq 1$

$\log_a b = \dfrac{1}{\log_b a}$ or $\log_a b \cdot \log_b a = 1$

Inverse Trigonometric Functions

1. Principal values & Domains of Inverse Trigonometric/Circular Functions :

	Function		Domain	Range
i.	$y = \sin^{-1} x$	where	$-1 \leq x \leq 1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$
ii.	$y = \cos^{-1} x$	where	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
iii.	$y = \tan^{-1} x$	where	$x \in R$	$-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$
iv.	$y = \text{cosec}^{-1} x$	where	$x \leq -1$ or $x \geq 1$	$-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}, y \neq 0$
v.	$y = \sec^{-1} x$	where	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi; y \neq \dfrac{\pi}{2}$
vi.	$y = \cot^{-1} x$	where	$x \in R$	$0 < y < \pi$

P -2

i. $\sin^{-1}(\sin x) = x, \ -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$
ii. $\cos^{-1}(\cos x) = x; \ 0 \leq x \leq \pi$
iii. $\tan^{-1}(\tan x) = x; \ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
iv. $\cot^{-1}(\cot x) = x; \quad 0 < x < \pi$
v. $\sec^{-1}(\sec x) = x; \ 0 \leq x \leq \pi, x \neq \dfrac{\pi}{2}$
vi. $\text{cosec}^{-1}(\text{cosec}\, x) = x; \ x \neq 0, -\dfrac{\pi}{2} \leq x < \dfrac{\pi}{2}$

P – 3

i. $\sin^{-1}(-x) = \sin^{-1} x, \quad -1 \leq x \leq 1$
ii. $\tan^{-1}(-x) = -\tan^{-1} x, \quad x \in R$
iii. $\cos^{-1}(-x) = \pi - \cos^{-1} x, \ -1 \leq x \leq 1$

iv. $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

P – 5

i. $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}, -1 \le x \le 1$

ii. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2},\ x \in R$

iii. $\operatorname{cosec}^{-1}x + \sec^{-1} x = \frac{\pi}{2},\ |x| \ge 1$

2. Identities of Addition and Substraction :

I-1

i. $\sin^{-1}x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right],\ x \ge 0, y \ge 0 \,\&\, (x^2+y^2) \le 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], x \ge 0, y \ge 0 \,\&\, x^2+y^2 > 1$

ii. $\cos^{-1}x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right], x \ge 0, y \ge 0$

iii. $\tan^{-1}x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0 \,\&\, xy < 1$

$= \pi + \tan^{-1}\frac{x+y}{1-xy}, x > 0, y > 0 \,\&\, xy > 1 = \frac{\pi}{2}, x > 0, y > 0 \,\&\, xy = 1$

I-2

i. $\sin^{-1}x - \sin^{-1} x = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right], x \ge 0, y \ge 0$

ii. $\cos^{-1}x + \cos^{-1} y = \cos^{-1}\left[xy - \sqrt{1-x^2}\ \sqrt{1-y^2}\right], x \ge 0, y \ge 0, x \le y$

iii. $\tan^{-1}x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}, x \ge 0, y \ge 0$

I-3

i. $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} 2\sin^{-1} & \text{if } |x| \le \frac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & \text{if } x > \frac{1}{\sqrt{2}} \\ -\left(\pi + 2\sin^{-1} x\right) & \text{if } x < -\frac{1}{\sqrt{2}} \end{cases}$

ii. $\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1} x \text{ if } 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x \text{ if } -1 \le x < 0 \end{cases}$

iii. $\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1} x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & \text{if } x > 1 \end{cases}$

iv. $\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1} x \text{ if } |x| \le 1 \\ \pi - 2\tan^{-1}x \text{ if } x > 1 \\ -\left(\pi + 2\tan^{-1} x\right) \text{ if } x < -1 \end{cases}$

v. $\cos^{-1}\dfrac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \geq 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$

$\text{if } \tan^{-1} + x\tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, x > 0,y > 0,z > 0 & (xy+yz+zx) < 1

NOTE:

i. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

ii. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$

iii. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

iv. $\tan^{1} 1 + \tan^{1}\dfrac{1}{2} + \tan^{1}\dfrac{1}{3} = \dfrac{\pi}{2}$

Solution of Triangle

1. Sine Rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

2. Cosine Formula :

i. $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

ii. $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

iii. $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

3. Projection Formula :

i. $a = b\cos C + c\cos B$

ii. $b = c\cos A + a\cos C$

iii. $c = a\cos B + b\cos A$

4. Napier's Analogy – tangent rule :

i. $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

ii. $\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

iii. $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

5. Trigonometric Functions of Half Angles :

i. $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$; $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

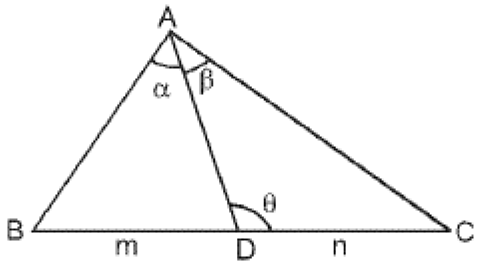
ii. $\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$; $\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ca}}$; $\cos\dfrac{C}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$

iii. $\tan\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\Delta}{s(s-a)}$ where $s = \dfrac{a+b+c}{2}$ is semi perimetre of triangle

iv. $\sin A = \dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \dfrac{2\Delta}{bc}$

6. **Area of Triangle (Δ) :** $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \sqrt{s(s-a)(s-b)(s-c)}$

7. **M – n Rule :**
If BD: DC = m : n, then
$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
$= n\cot B - m\cot C$



8. **Radius of Circumcircle :**
$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

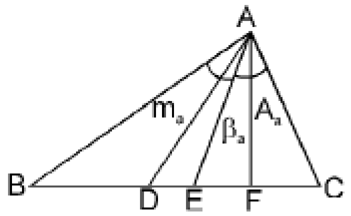
9. **Radius of The Incircle:**
   i. $r = \frac{\Delta}{s}$
   ii. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$
   iii. $r = \dfrac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on
   iv. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. **Radius of The Ex-Circles :**
   i. $r_1 = \frac{\Delta}{s-a} : r_2 = \frac{\Delta}{s-b} : r_3 = \frac{\Delta}{s-c}$
   ii. $r_1 = s\tan\frac{A}{2} : r_2 = s\tan\frac{B}{2} : r_3 = s\tan\frac{C}{2}$
   iii. $r_1 = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on
   iv. $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

11. **Length of Angle Bisectors, Medians & Altitudes :**
   i. Length of an angle bisector from the angle A = $\beta_a = \dfrac{2bc\cos\frac{A}{2}}{b+c}$;

ii. Length of median from the angle A = $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

iii. Length of altitude from the angle A = $A_a = \frac{2\Delta}{a}$



12. The Distances of the Special Points from Vertices and Sides of Triangle :

i. Circumcentre (O) : OA = R & O_a = R cos A

ii. Incentre (I) : IA = r cosec $\frac{A}{2}$ & $l_a = r$

iii. Excentre (l_1): l_1A = $r_1 = \operatorname{cosec}\frac{A}{2}$

iv. Orthocenter: HA = 2R cos A & H_a = 2R cos B cos C

v. Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$ & $G_a = \frac{2\Delta}{3a}$

13. Orthocentre and Pedal Triangle :

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.

i. Its angle are π – 2A, π – 2B and π – 2C

ii. Its sides are a cos A = R sin 2A.
b cos B = R sin 2B and
c cos C = R sin 2C

iii. Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

14. Excentral Triangle :

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

i. ΔABC is the pedal triangle of the Δ $I_1I_2I_3$.

ii. its angles are $\frac{\pi}{2} - \frac{A}{2}, \frac{\pi}{2} - \frac{B}{2} \& \frac{\pi}{2} - \frac{C}{2}$.

iii. its sides are 4R cos $\frac{A}{2}$, 4 R cos $\frac{B}{2}$ & 4 R cos $\frac{C}{2}$

iv. $I\,I_1 = 4R \sin\frac{A}{2}$; $I\,I_2 = 4R \sin\frac{B}{2}$; $I\,I_3 = 4R \sin\frac{C}{2}$

v. Incentre I of ΔABC is the orthocenter of the excentral $I_1I_2I_3$

15. Distance Between Special Points :

i. Distance between circumcentre and orthocenter $OH^2 = R^2(1 - 8 \cos A \cos B \cos C)$

ii. Distance between circumcentre and incentre $OI^2 = R^2(1 - 8 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}) = R^2 - 2Rr$

iii. Distance between circumcentre and centroid $OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$

FORMULAE

1. A mathematically acceptable statement is a sentence which is either true or false.
2. Negation of a statement p: If p denote a statement, then the negation of p is denoted by $\sim p$.
3. Compound statements and their related component statements:
   A statement is a compound statement if it is made up of two or more smaller statements. The smaller statements are called component statements of the compound statement.
4. The role of "And", "Or", "There exists" and "For every" in compound statements.
5. The meaning of implications "If ", "only if ", " if and only if ".
   A sentence with if p, then q can be written in the following ways.
6. p implies q (denoted by $p \Rightarrow q$)
7. p is a sufficient condition for q
8. q is a necessary condition for p
9. p only if q
10. $\sim q$ implies $\sim p$
11. The contrapositive of a statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$. The converse of a statement $p \Rightarrow q$ is the statement $q \Rightarrow p$. $p \Rightarrow q$ together with its converse, gives p if and only if q.
12. The following methods are used to check the validity of statements:
    i. direct method
    ii. contrapositive method
    iii. method of contradiction
    iv. using a counter example.